Asymptotic of implicit functions if $F_{ww} = 0$

(*Václav Kotěšovec, published 19.1.2014*)

Main result:

Theorem (V. Kotěšovec, 2013)
With same notation and same conditions as in theorem by Bender (see below), but if $F_{ww} = 0$ and $F_{www} \neq 0$ then

$$a_n \sim \frac{1}{3\ \Gamma\left(\frac{2}{3}\right)\ n\ r^n} * \left(-\frac{6\ r\ F_z}{n\ F_{www}}\right)^{1/3}$$

For exponential generating function is

$$a_n \sim \frac{\Gamma\left(\frac{1}{3}\right)\ n^{n-5/6}}{6^{1/6}\sqrt{\pi}\ e^n\ r^{n-1/3}} * \left(-\frac{F_z}{F_{www}}\right)^{1/3}$$

where Γ is the Gamma function
r is the radius of convergence
F_z is partial derivative of the function $F(z,w)$ at the point $[r,s]$
F_{www} is third partial derivative of the function $F(z,w)$ at the point $[r,s]$

Following theorem by Edward A. Bender is very useful, but is not possible apply it, if second partial derivative

$$F_{ww}(r,s) = 0$$

is zero.

Citation: Edward A. Bender, "Asymptotic methods in enumeration" (1974), p.502, see [1]

THEOREM 5. *Assume that the power series* $w(z) = \sum a_n z^n$ *with nonnegative coefficients satisfies* $F(z, w) \equiv 0$. *Suppose there exist real numbers* $r > 0$ *and* $s > a_0$ *such that*

(i) *for some* $\delta > 0$, $F(z, w)$ *is analytic whenever* $|z| < r + \delta$ *and* $|q| < s + \delta$;
(ii) $F(r, s) = F_w(r, s) = 0$;
(iii) $F_z(r, s) \neq 0$, *and* $F_{ww}(r, s) \neq 0$: *and*
(iv) *if* $|z| \leqq r, |w| \leqq s$, *and* $F(z, w) = F_w(z, w) = 0$, *then* $z = r$ *and* $w = s$.

Then

(7.1) $$a_n \sim ((rF_z)/(2\pi F_{ww}))^{1/2} n^{-3/2} r^{-n},$$

where the partial derivatives F_z *and* F_{ww} *are evaluated at* $z = r$, $w = s$.

For proof see [1], p.505 and also [2], p.469.

7.3. Proof of Theorem 5. In the region where F is analytic, all singularities of $w(z)$ are determined by $F_w = 0$. Since $a_n \geqq 0$, $|w(z)| \leqq w(|z|)$. From this and the hypotheses it is clear that the radius of convergence of the power series for $w(z)$ is r and that there is only one singularity on the boundary, namely at $z = r$. By the Weierstrass preparation theorem (see Markushevich [35, pp. 105–112]), in a neighborhood of (r, s), F behaves like

(7.9) $$F + (w - s)F_w + (z - r)F_z + (w - s)^2 F_{ww}/2,$$

where F and its derivatives are evaluated at the point (r, s). Since $F = F_w = 0$, the zeros of (7.9) are

$$s \pm (2(r - z)F_z/F_{ww})^{1/2}.$$

Thus the singularity is a branch point and (7.1) follows from Theorem 4. Furthermore $w(z)$, viewed as a solution of $F(z, w) = 0$, is well-defined for $z = r$: in fact, $w(r) = s$.

If $F_{ww} = 0$, then we must add term F_{www}. Taylor series in two variables see [3]. Smaller order terms can be ignored and we have

$$(z - r) * F_z + (w - s)^3 * \frac{F_{www}}{6} = 0$$

$$w - s = \left(6\ F_z * \frac{r - z}{F_{www}}\right)^{1/3}$$

Now, final asymptotic follows from following Darboux's theorem

Citation: Edward A. Bender, "Asymptotic methods in enumeration" (1974), p.498

> **6. Functions well-behaved on the circle of convergence.** Suppose the function f has a singularity at α. The singularity is called *algebraic* if $f(z)$ can be written as a function analytic near α plus a finite sum of terms of the form
>
> (6.1) $$(1 - z/\alpha)^{-\omega} g(z),$$
>
> where g is a function which is analytic and nonzero near α and ω is a complex number not equal to $0, -1, -2, \cdots$. The *weight* of (6.1) is the real part of ω. The following special case of Darboux's theorem (see Szegö [53, p. 205]) is convenient for handling many generating functions which are well-behaved on the circle of convergence.
>
> THEOREM 4. *Suppose $A(z) = \sum_{n \geq 0} a_n z^n$ is analytic near 0 and has only algebraic singularities on its circle of convergence. Let w be the maximum of the weights at these singularities. Denote by α_k, ω_k and g_k the values of α, ω and g for those terms of the form* (6.1) *of weight w. Then*
>
> (6.2) $$a_n - \frac{1}{n} \sum_k \frac{g_k(\alpha_k) n^{\omega_k}}{\Gamma(\omega_k) \alpha_k^n} = o(r^{-n} n^{w-1}),$$
>
> *where $r = |\alpha_k|$, the radius of convergence of $A(z)$, and $\Gamma(s)$ is the gamma function.*

In our case is $k = 1,$ and $\alpha = r$

$$w - s = \left(6\, F_z \,\frac{r - z}{F_{www}}\right)^{1/3} = \left(1 - \frac{z}{r}\right)^{1/3} * \left(6\, F_z \,\frac{r}{F_{www}}\right)^{1/3}$$

$$\omega = -\frac{1}{3}$$

$$g(r) = \left(6\, F_z \,\frac{r}{F_{www}}\right)^{1/3}$$

Identities for Gamma function:

$$-\frac{1}{\Gamma\left(-\frac{1}{3}\right)} = \frac{1}{3\,\Gamma\left(\frac{2}{3}\right)} = \frac{\Gamma\left(\frac{1}{3}\right)}{2\pi\sqrt{3}} = 0.24616270387$$

For **nonnegative** coefficients a_n is final asymptotic following:

$$a_n \sim \frac{1}{n} * \left(6\, F_z \,\frac{r}{F_{www}}\right)^{1/3} * \frac{n^{-1/3}}{\Gamma\left(-\frac{1}{3}\right) * r^n} = \frac{1}{3\,\Gamma\left(\frac{2}{3}\right)\, n\, r^n} * \left(-\frac{6\, r\, F_z}{n\, F_{www}}\right)^{1/3}$$

In case of exponential generating function (with help of Stirling's formula)

$$a_n \sim \frac{\Gamma\left(\frac{1}{3}\right)\, n^{n-5/6}}{6^{1/6}\sqrt{\pi}\, e^n\, r^{n-1/3}} * \left(-\frac{F_z}{F_{www}}\right)^{1/3}$$

Remark: In comparison with original Bender's formula for case $F_{ww} \neq 0$ (modified for exponential generating function)

$$\frac{a_n}{n!} \sim \frac{1}{n\, r^n}\sqrt{\frac{r\, F_z}{2\pi n\, F_{ww}}}$$

$$a_n \sim \frac{n^{n-1}}{e^n\, r^{n-1/2}}\sqrt{\frac{F_z}{F_{ww}}}$$

is difference not only in constant term, but also

$$n^{1/6}$$

Applications - sequences from the OEIS

A200317

$$f(x,y) = x - y - \cos(y) + 1$$

System of the equations:

$r + 1 = s + cos(s)$	$sin(s) = 1$
$r = \frac{\pi}{2} - 1$	$s = \frac{\pi}{2}$

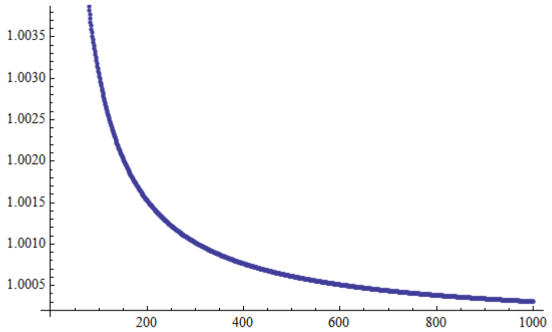
partial derivatives			at the point [r,s]
F_z	$\frac{\partial}{\partial x} f(x,y)$	1	1
F_w	$\frac{\partial}{\partial y} f(x,y)$	$\sin(y) - 1$	0
F_{ww}	$\frac{\partial}{\partial y}\frac{\partial}{\partial y} f(x,y)$	$\cos(y)$	**0**
F_{www}	$\frac{\partial}{\partial y}\frac{\partial}{\partial y}\frac{\partial}{\partial y} f(x,y)$	$-\sin(y)$	-1

Asymptotic:

$$a_n \sim \frac{\Gamma\left(\frac{1}{3}\right) n^{n-5/6}}{6^{1/6} \sqrt{\pi}\, e^n \left(\frac{\pi}{2} - 1\right)^{n-1/3}}$$

Verification:

```
ListPlot[Table[A200317[[n]] / ((e^-n n^(-5/6+n) (-1 + π/2)^(1/3-n) Gamma[1/3]) / (6^(1/6) Sqrt[π])), {n, 1, 1000}]]
```




```
funs[n_] := A200317[[n]] / ((e^-n n^(-5/6+n) (-1 + π/2)^(1/3-n) Gamma[1/3]) / (6^(1/6) Sqrt[π]));

Do[
 Print[
  N[Sum[(-1)^(m + j) * funs[j * Floor[Length[A200317] / m]] *
     j^(m - 1) / (j - 1)! / (m - j)!, {j, 1, m}], 40]], {m, 1, 51, 10}]
1.000305042741662857527933906742033397601
1.000000017153570245900387311879066373242
1.000000007018654665455701969073283177878
1.000000004098224498481834306719168905208
1.000000002836403602844345666835915580867
1.000000002156273181428198061215851902436
```

Richardson extrapolation used for 1000 terms and convergence is good.

A143134

$$f(x,y) = x - y + \sin^2(y)$$

System of the equations:

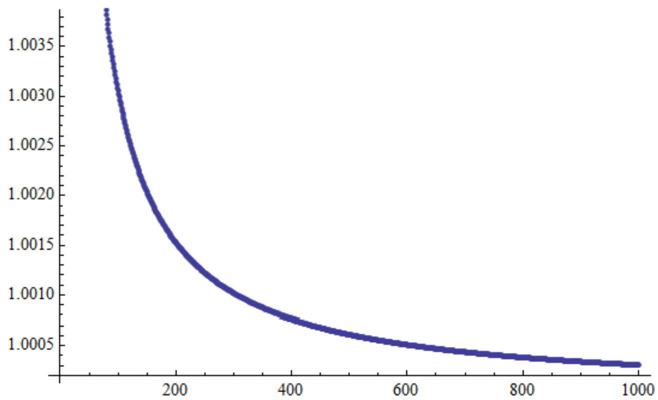
$\sin^2(s) + r = s$	$\sin(2s) = 1$
$r = \frac{\pi}{4} - \frac{1}{2}$	$s = \frac{\pi}{4}$

partial derivatives			at the point [*r*,*s*]
F_z	$\frac{\partial}{\partial x} f(x,y)$	1	1
F_w	$\frac{\partial}{\partial y} f(x,y)$	$2\cos(y)\sin(y) - 1$	0
F_{ww}	$\frac{\partial}{\partial y}\frac{\partial}{\partial y} f(x,y)$	$2\cos^2(y) - 2\sin^2(y)$	**0**
F_{www}	$\frac{\partial}{\partial y}\frac{\partial}{\partial y}\frac{\partial}{\partial y} f(x,y)$	$-8\cos(y)\sin(y)$	-4

Asymptotic:

$$a_n \sim \frac{\Gamma\left(\frac{1}{3}\right)\ 2^{2*n-3/2}\ n^{n-5/6}}{3^{1/6}\sqrt{\pi}\ e^n\ (\pi-2)^{n-1/3}}$$

$$\text{ListPlot}\left[\text{Table}\left[\text{A143134}[[n]] \middle/ \left(\frac{2^{-\frac{3}{2}+2n}\ e^{-n}\ n^{-\frac{5}{6}+n}\ (-2+\pi)^{\frac{1}{3}-n}\ \text{Gamma}\left[\frac{1}{3}\right]}{3^{1/6}\sqrt{\pi}}\right), \{n, 1, 1000\}\right]\right]$$



A143135

$$f(x,y) \; = \; \sin(x+y^2)-y$$

System of the equations:

$s=\sin(s^2+r)$	$2s\cos(s^2+r)=1$
$r \; = \dfrac{\pi}{4}-\dfrac{1}{2}$	$s=\dfrac{1}{\sqrt{2}}$

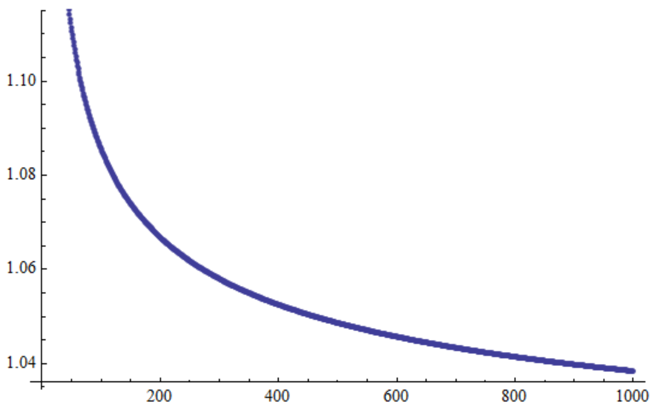
partial derivatives			at the point [*r*,*s*]
F_z	$\dfrac{\partial}{\partial x}f(x,y)$	$\cos(y^2+x)$	$\dfrac{1}{\sqrt{2}}$
F_w	$\dfrac{\partial}{\partial y}f(x,y)$	$2\,y\cos(y^2+x)-1$	0
F_{ww}	$\dfrac{\partial}{\partial y}\dfrac{\partial}{\partial y}f(x,y)$	$2(\cos(y^2+x)-2y^2\sin(y^2+x))$	**0**
F_{www}	$\dfrac{\partial}{\partial y}\dfrac{\partial}{\partial y}\dfrac{\partial}{\partial y}f(x,y)$	$-4y(2\cos(y^2+x)\,y^2+3\sin(y^2+x))$	-8

Asymptotic:

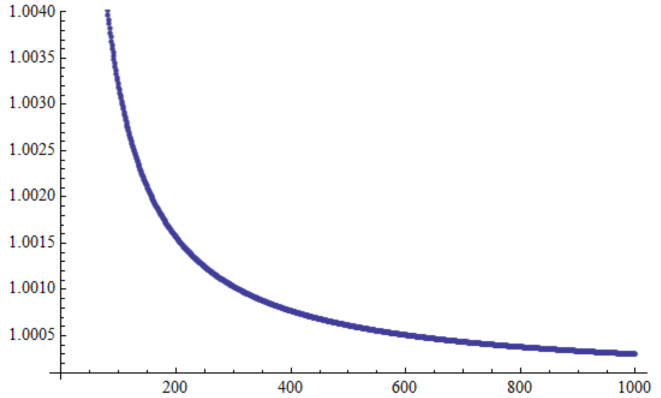
$$a_n \;\sim\; \frac{\Gamma\left(\frac{1}{3}\right)\;4^{n-1}\,n^{n-5/6}}{3^{1/6}\sqrt{\pi}\,e^n\,(\pi-2)^{n-1/3}} * \left(1+\frac{0.3810\ldots}{n^{1/3}}+\cdots\right)$$

In this case is convergence with 1000 terms slow, therefore I added minor asymptotic term yet (computed numerically).

$$\text{ListPlot}\left[\text{Table}\left[\text{A143135}[[n]]\Big/\left(\frac{4^{-1+n}\,e^{-n}\,n^{-\frac{5}{6}+n}\,(-2+\pi)^{\frac{1}{3}-n}\,\text{Gamma}\left[\frac{1}{3}\right]}{3^{1/6}\sqrt{\pi}}\right),\ \{n,\,1,\,1000\}\right]\right]$$



$$\text{ListPlot}\left[\text{Table}\left[\text{A143135}[[n]]\Big/\left(\frac{4^{-1+n}\,e^{-n}\,n^{-\frac{5}{6}+n}\,(-2+\pi)^{\frac{1}{3}-n}\,\text{Gamma}\left[\frac{1}{3}\right]}{3^{1/6}\sqrt{\pi}} * \left(1+\frac{0.381`}{n^{1/3}}\right)\right),\ \{n,\,1,\,1000\}\right]\right]$$



A218652

$$f(x,y) = x + log(y^2+1) - y$$

System of the equations:

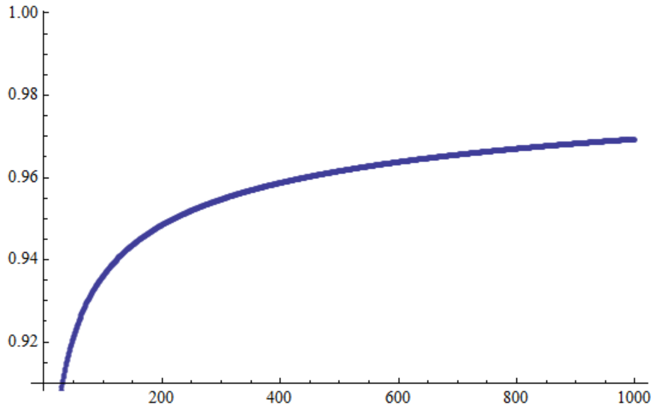
$r + \log(s^2+1) = s$	$\frac{2s}{s^2+1} = 1$
$r = 1 - \log(2)$	$s = 1$

partial derivatives			at the point [*r*,*s*]
F_z	$\frac{\partial}{\partial x} f(x,y)$	1	1
F_w	$\frac{\partial}{\partial y} f(x,y)$	$\frac{2y}{y^2+1} - 1$	0
F_{ww}	$\frac{\partial}{\partial y}\frac{\partial}{\partial y} f(x,y)$	$-\frac{2(y^2-1)}{(y^2+1)^2}$	**0**
F_{www}	$\frac{\partial}{\partial y}\frac{\partial}{\partial y}\frac{\partial}{\partial y} f(x,y)$	$\frac{4y(y^2-3)}{(y^2+1)^3}$	-1

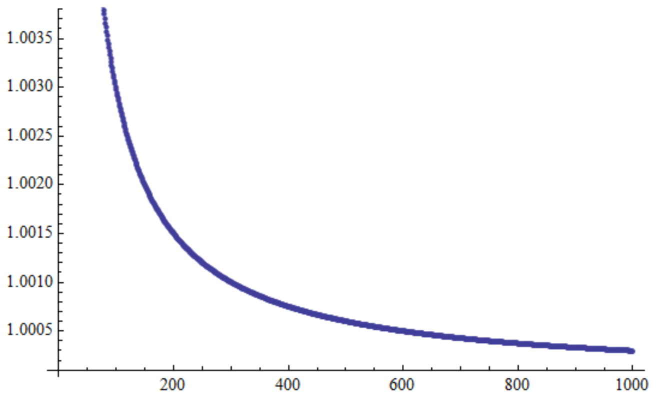
Asymptotic:

$$a_n \sim \frac{\Gamma\left(\frac{1}{3}\right) n^{n-5/6}}{6^{1/6}\sqrt{\pi}\, e^n \left(1 - log(2)\right)^{n-1/3}} * \left(1 - \frac{0.3096 \ldots}{n^{1/3}} + \cdots\right)$$

```
ListPlot[
 Table[A218652[[n]]/
  (e^-n n^(-5/6+n) Gamma[1/3] (1 - Log[2])^(1/3-n) / (6^(1/6) Sqrt[π])), {n, 1, 1000}]]
```




```
ListPlot[
 Table[A218652[[n]]/
  (e^-n n^(-5/6+n) Gamma[1/3] (1 - Log[2])^(1/3-n) / (6^(1/6) Sqrt[π]) * (1 - 0.3096`/n^(1/3))),
  {n, 1, 1000}]]
```



Czech abstract:

Pokud není funkce definována obvyklým způsobem jako y=f(x), ale implicitně jako f(x,y)=0, lze asymptotický průběh koeficientů a_n mocninné řady, dané touto vytvořující funkcí f, určit pomocí věty z článku Edwarda Bendera (1974), viz [1]. Jedním z předpokladů této věty však je, aby v bodu extrému [r,s] (který je určen soustavou 2 rovnic) byla druhá parciální derivace podle y nenulová. Pokud tato podmínka není splněna, dělilo by se v jeho vzorci nulou a vzorec proto nejde v takovém případě použít.
Můj článek se zabývá právě tímto případem, kdy je druhá parciální derivace, označovaná jako F_{ww} rovna nule, ale současně třetí parciální derivace F_{www} je různá od nuly (případně by bylo možné zobecnění i na první nenulovou derivaci vyššího řádu). V tomto případě jsem pomocí vybraných členů Taylorova rozvoje pro funkce 2 proměnných nejprve určil chování funkce v okolí bodu [r,s] a potom odtud odvodil pro hodnoty (nezáporných) koeficientů a_n asymptotický vzorec pro tento případ. K důkazu mojí věty jsem použil, stejně jako Bender, ještě jednu pomocnou větu, jejímž autorem je Darboux. Výsledkem je, že koeficienty sekvence rostou řádově o člen $n^{1/6}$ rychleji než v případech, kdy byla druhá parciální derivace nenulová.
Ve druhé části článku je věta aplikována na několik sekvencí z OEIS, odvozeny asymptotické vzorce a provedena numerická verifikace pro prvních 1000 členů těchto posloupností. K tomu je třeba poznamenat, že vygenerování 1000 členů každé z těchto sekvencí trvalo programem Mathematica přes 30 hodin.
Metoda rozšiřuje oblast sekvencí, pro které lze analyticky nalézt asymptotický průběh, který by se jinak hledal velmi obtížně nebo by byl možný jen numerický odhad.

References:

[1] Edward A. Bender, Asymptotic methods in enumeration, SIAM Review 16 (1974), no. 4, 485-515
[2] P. Flajolet and R. Sedgewick, Analytic Combinatorics, 2009, p. 469
[3] Wikipedia, Taylor series in several variables
[4] OEIS - The On-Line Encyclopedia of Integer Sequences
[5] Kotěšovec V., Interesting asymptotic formulas for binomial sums, website 9.6.2013

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