Asymptotic of sequence A084611

(*Václav Kotěšovec, published 26.7.2013*)

In the OEIS (On-Line Encyclopedia of Integer Sequences) published Paul D. Hanna 13.9.2003 sequence A084611, a_n = sum of absolute values of coefficients of $(1+x-x^2)^n$ and 12.7.2013 he conjectured asymptotic

$$\lim_{n\to\infty} (a_n)^{1/n} \sim \sqrt{5}$$

This article contains proof of this formula.

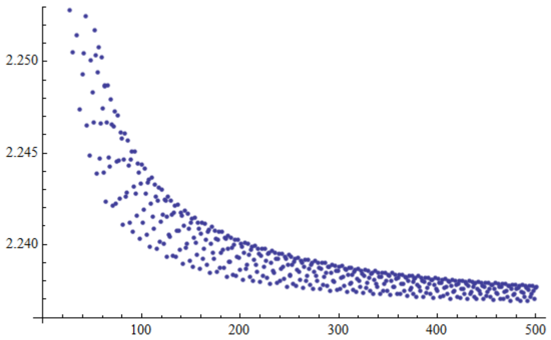
First several terms of this sequence is (in program Mathematica)

```
Table[Sum[Abs[Coefficient[Expand[(1+x-x^2)^n],x,k]],{k,0,2*n}],{n,0,20}]
```

{1, 3, 7, 13, 35, 83, 165, 367, 899, 1957, 3839, 9771, 22709, 43213, 102963, 255061, 525601, 1098339, 2798273, 6202969, 11746259}

Numerical results for first 500 terms are

```
ListPlot[Table[A084611[[n]]^(1/n), {n, 1, 500}]]
```

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Sqrt[5.]
2.23607
```

This polynomial $1+x-x^2$ has two real roots

$$x_1 = \frac{1}{2}(1+\sqrt{5}), \qquad x_2 = \frac{1}{2}(1-\sqrt{5})$$

With help of binomial theorem and convolution of $(x-x_1)^n * (x-x_2)^n$ we obtain following result

$$\left(\sum_{k=0}^{n}\binom{n}{k} * (-x_1)^k * x^{n-k}\right) * \left(\sum_{m=0}^{n}\binom{n}{m} * (-x_2)^m * x^{n-m}\right) = \sum_{p=0}^{2n}\sum_{k=0}^{p}(-x_1)^{n-k}\binom{n}{k}(-x_2)^{k+n-p}\binom{n}{p-k} * x^p$$

Now is the sum of absolute values of coefficients (for $x=1$)

```
alfa = (1 + Sqrt[5]) / 2; beta = (Sqrt[5] - 1) / 2;
Quiet[
 FullSimplify[
  Table[
   Sum[Abs[Sum[Binomial[n, k] * Binomial[n, p - k] * (-alfa) ^ (n - k) * beta ^ (n - p + k),
      {k, 0, p}]], {p, 0, 2 n}], {n, 0, 20}]]]
{1, 3, 7, 13, 35, 83, 165, 367, 899, 1957, 3839, 9771, 22709,
 43213, 102963, 255061, 525601, 1098339, 2798273, 6202969, 11746259}
```

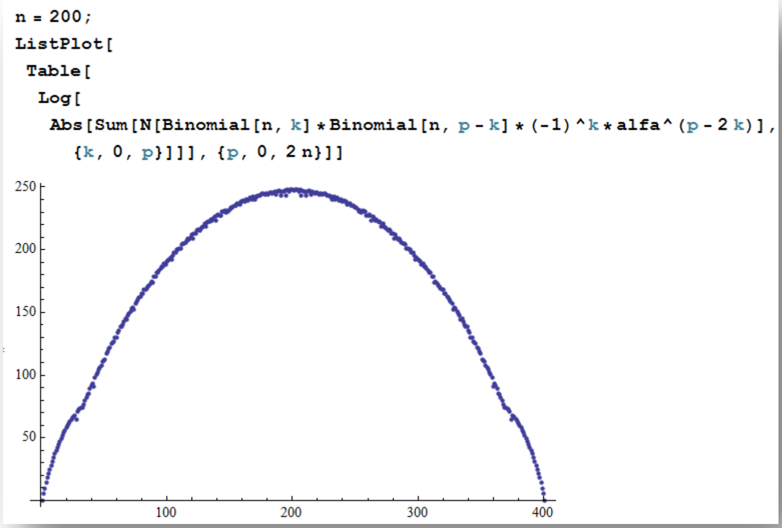
Simplified to

$$a_n = \sum_{p=0}^{2n}\left|\sum_{k=0}^{n}(-1)^k\left(\frac{\sqrt{5}+1}{2}\right)^{p-2k}\binom{n}{k}\binom{n}{p-k}\right|$$

Maximal term of outer sum is at position

$$p = n$$

and function is symmetrical, see following graph



Now, if we search main asymptotic term only, is sufficient (for purpose of this proof) find the value at the maximum and then

$$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} (2n * a_{nmax})^{1/n}$$

For $p = n$ we have sum

$$\sum_{k=0}^{n} (-1)^k \left(\frac{\sqrt{5}+1}{2}\right)^{n-2k} \binom{n}{k}\binom{n}{n-k} = \sum_{k=0}^{n} (-1)^k \left(\frac{\sqrt{5}+1}{2}\right)^{n-2k} \binom{n}{k}^2$$

Generally, sum

$$\sum_{k=0}^{n} (-1)^k z^k \binom{n}{k}^2 = (1+z)^n * P_n\left(\frac{1-z}{1+z}\right)$$

where P_n is Legendre polynomial.
For non-alternating sum see [3]. Note, that generating function for this sum is

$$\frac{1}{\sqrt{1+(2z-2)x+(z+1)^2x^2}}$$

In our case is

$$z = \frac{1}{\left(\frac{\sqrt{5}+1}{2}\right)^2}$$

and

$$\sum_{k=0}^{n} (-1)^k \left(\frac{\sqrt{5}+1}{2}\right)^{n-2k} \binom{n}{k}^2 = \left(\frac{\sqrt{5}+1}{2}\right)^n (1+z)^n * P_n\left(\frac{1}{\sqrt{5}}\right)$$

Asymptotic of Legendre polynomials is already know, see for example [5].
For $x > 1$ and $n \to \infty$

$$P_n(x) \sim \frac{\left(\sqrt{x^2-1}+x\right)^{n+\frac{1}{2}}}{\sqrt{2\pi n}\,\sqrt[4]{x^2-1}}$$

For small arguments (our case)

$$P_n(\cos(x)) \sim \sqrt{\frac{2}{\pi\, n \sin(x)}} * \cos\left(\left(n+\frac{1}{2}\right)x - \frac{\pi}{4}\right)$$

and

$$\lim_{n\to\infty}\left(P_n\left(\frac{1}{\sqrt{5}}\right)\right)^{1/n} = 1$$

Finally

$$\lim_{n\to\infty}(a_n)^{1/n} = \lim_{n\to\infty}\left(\left(\frac{\sqrt{5}+1}{2}\right)^n * (1+z)^n * P_n\left(\frac{1}{\sqrt{5}}\right)\right)^{1/n} = \left(\frac{\sqrt{5}+1}{2}\right) * \left(1+\frac{1}{\left(\frac{\sqrt{5}+1}{2}\right)^2}\right) = \sqrt{5}$$

QED.

References:

[1] OEIS - The On-Line Encyclopedia of Integer Sequences
[2] Kotěšovec V., Interesting asymptotic formulas for binomial sums, website 9.6.2013
[3] Kotěšovec V., Asymptotic of a sums of powers of binomial coefficients * x^k, website 20.9.2012
[4] Kotěšovec V., Asymptotic of generalized Apéry sequences with powers of binomial coefficients, website 4.11.2012
[5] F. W. J. Olver, "Asymptotics and Special Functions", Academic Press, New York, 1974.

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