Asymptotic of sequences A244820, A244821 and A244822

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In the OEIS (On-Line Encyclopedia of Integer Sequences) published Paul D. Hanna 6.7.2014 sequences A244820, A244821 and A244822, which can be generalized as

$$a_n = \sum_{k=0}^{n} p^{k(n-k)}\, k^{n-k} \binom{n}{k}$$

where p is integer > 1.

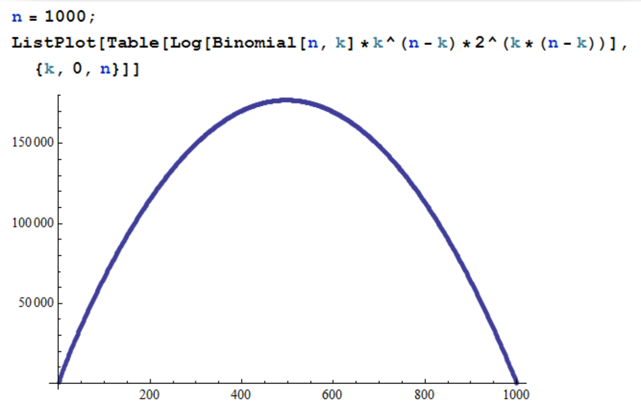
Main result:

$$\sum_{k=0}^{n} p^{k(n-k)}\, k^{n-k} \binom{n}{k} \sim \frac{1}{\sqrt{\pi}} * e^{\frac{(1+\log(2))^2}{4\log(p)}} * 2^{\frac{n+1}{2}} * n^{\frac{n-1}{2}+\frac{\log(n)}{4\log(p)}-\frac{1+\log(2)}{2\log(p)}} * p^{\frac{n^2}{4}} * \sum_{m=-\infty}^{\infty} p^{-\left(m-frac\left(\frac{n}{2}-\frac{\log\left(\frac{n}{2}\right)-1}{2\log(p)}\right)\right)^2}$$

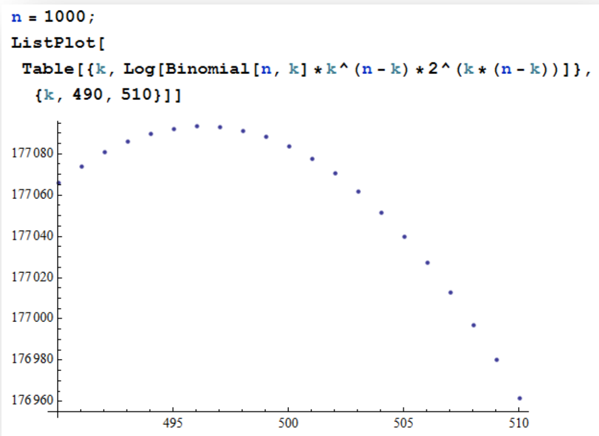
where "*frac*" is the fractional part.

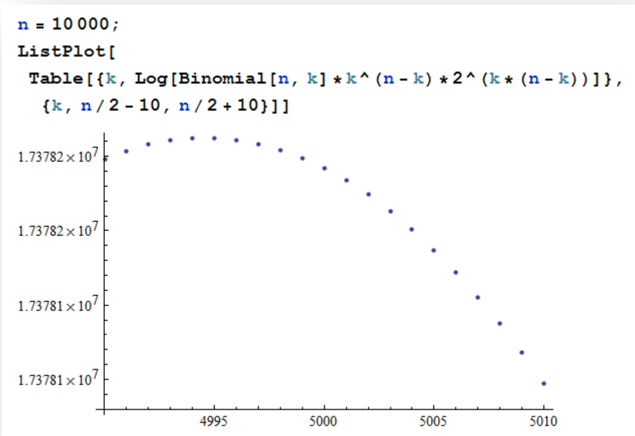
$$\sum_{m=-\infty}^{\infty} p^{-\left(m+\frac{1}{2}\right)^2} \le \sum_{m=-\infty}^{\infty} p^{-\left(m-frac\left(\frac{n}{2}-\frac{\log\left(\frac{n}{2}\right)-1}{2\log(p)}\right)\right)^2} \le \sum_{m=-\infty}^{\infty} p^{-m^2}$$

For first orientation here is graph for $p = 2$ in the logarithmical scale:



The maximal term is at position near $n/2$, but not exactly at $n/2$.





We find the **maximal term** with the help of Stirling's formula, derivative must be zero

```
stirling[n_] := n^n / E^n * Sqrt[2 * Pi * n];
binom[n_, k_] := stirling[n] / stirling[k] / stirling[n - k];

FullSimplify[D[binom[n, n / 2 - m] * (n / 2 - m) ^ (n / 2 + m) * p ^ ((n / 2 - m) * (n / 2 + m)), m]]
```

$$-\frac{1}{(2\,m+n)^{3/2}\sqrt{\pi}}\,2^{\frac{1}{2}-2\,m}\left(m+\frac{n}{2}\right)^{-m-\frac{n}{2}} n^{\frac{1}{2}+n}\,(-2\,m+n)^{-\frac{3}{2}+2\,m}\,p^{\frac{1}{4}\left(-4\,m^2+n^2\right)}\Big(4\,(-1+m)\,m+4\,m\,n+n^2+\left(-4\,m^2+n^2\right)\mathrm{Log}\left[m+\frac{n}{2}\right]-2\left(4\,m^2-n^2\right)\left(\mathrm{Log}[2]-\mathrm{Log}[-2\,m+n]+m\,\mathrm{Log}[p]\right)\Big)$$

Now we have

```
Limit[
 (4 (-1 + m) m + 4 m n + n^2 + (-4 m^2 + n^2) Log[m + n/2] - 2 (4 m^2 - n^2) (Log[2] - Log[-2 m + n] + m Log[p])) /
  n^2 / Log[n] /. m → c * Log[n], n → Infinity]
```

$-1+2\,c\,\mathrm{Log}[p]$

```
Solve[-1 + 2 c Log[p] == 0]
```

$\left\{\left\{c\to\frac{1}{2\,\mathrm{Log}[p]}\right\}\right\}$

and in next step

```
Limit[
 (4 (-1 + m) m + 4 m n + n^2 + (-4 m^2 + n^2) Log[m + n/2] - 2 (4 m^2 - n^2) (Log[2] - Log[-2 m + n] + m Log[p])) /
  n^2 /. m → 1/(2 Log[p]) * Log[n] + d, n → Infinity]
```

$1+\mathrm{Log}[2]+2\,d\,\mathrm{Log}[p]$

```
Solve[1 + Log[2] + 2 d Log[p] == 0]
```

$\left\{\left\{d\to\frac{-1-\mathrm{Log}[2]}{2\,\mathrm{Log}[p]}\right\}\right\}$

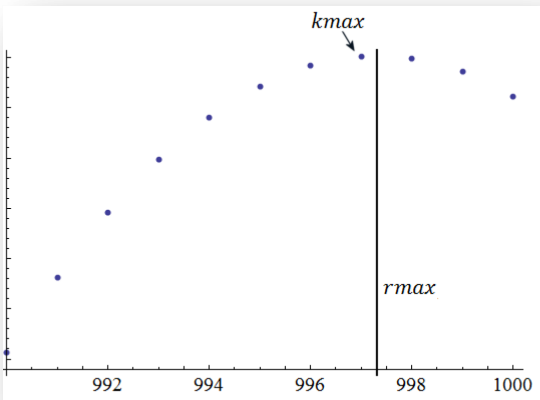
Maximum of this (real) function is at the point

$$rmax\ =\frac{n}{2}-\frac{\log(n)}{2\log(p)}+\frac{1+\log(2)}{2\log(p)}=\frac{n}{2}-\frac{\log(n/2)-1}{2\log(p)}$$

Complication is, that this point is (in general) **not integer**. Maximum term of the sequence is at the integer point nearest this real point.

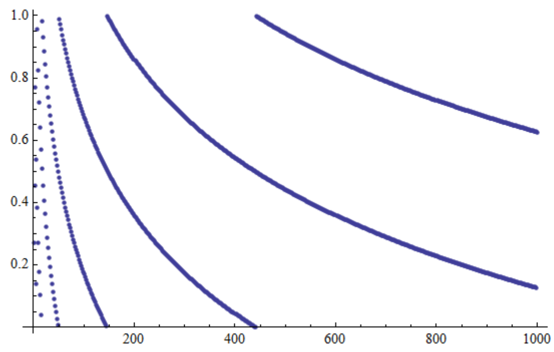
$$\text{if}\ \ frac(rmax)\le 1/2\ \ \text{then}\ kmax=floor(rmax)$$
$$\text{if}\ \ frac(rmax)>1/2\ \ \text{then}\ kmax=floor(rmax)+1$$

where "*frac*" is the fractional part.



Interesting is graph of distribution of fractional parts (example for $p = 3$)

```
p = 3; ListPlot[Table[FractionalPart[n / 2 - (Log[n / 2] - 1)/(2 Log[p])], {n, 1, 1000}]]
```



Value in the (real) maximum is then

$$a_{rmax} = \left(\frac{n}{2} - \frac{\log(n)}{2\log(p)} + \frac{1+\log(2)}{2\log(p)}\right)^{\frac{n}{2} + \frac{\log(n)}{2\log(p)} - \frac{1+\log(2)}{2\log(p)}} p^{\left(\frac{n}{2} - \frac{\log(n)}{2\log(p)} + \frac{1+\log(2)}{2\log(p)}\right)\left(\frac{n}{2} + \frac{\log(n)}{2\log(p)} - \frac{1+\log(2)}{2\log(p)}\right)} \binom{n}{\frac{n}{2} - \frac{\log(n)}{2\log(p)} + \frac{1+\log(2)}{2\log(p)}}$$

For purpose of asymptotic this expression can be simplified

$$\left(\frac{n}{2} - \frac{\log(n)}{2\log(p)} + \frac{1+\log(2)}{2\log(p)}\right)^{\frac{n}{2} + \frac{\log(n)}{2\log(p)} - \frac{1+\log(2)}{2\log(p)}} \sim e^{\frac{1+\log^2(2)}{2\log(p)}} 2^{\frac{1}{\log(p)} - \frac{n}{2}} n^{\frac{n\log(p) + \log\left(\frac{n}{4}\right) - 2}{2\log(p)}}$$

$$p^{\left(\frac{n}{2} - \frac{\log(n)}{2\log(p)} + \frac{1+\log(2)}{2\log(p)}\right)\left(\frac{n}{2} + \frac{\log(n)}{2\log(p)} - \frac{1+\log(2)}{2\log(p)}\right)} \sim p^{\frac{n^2}{4}} 2^{-\frac{1}{2\log(p)}} e^{-\frac{1+\log^2(2)}{4\log(p)}} n^{\frac{-\log(n)+2+\log(4)}{4\log(p)}}$$

$$\binom{n}{\frac{n}{2} - \frac{\log(n)}{2\log(p)} + \frac{1+\log(2)}{2\log(p)}} \sim \binom{n}{n/2} \sim \frac{2^{n+\frac{1}{2}}}{\sqrt{\pi n}}$$

and result is

$$a_{rmax} \sim \frac{1}{\sqrt{\pi}} * e^{\frac{(1+\log(2))^2}{4\log(p)}} * 2^{\frac{n+1}{2}} * n^{\frac{n-1}{2} + \frac{\log(n)}{4\log(p)} - \frac{1+\log(2)}{2\log(p)}} * p^{\frac{n^2}{4}}$$

Checked with Mathematica

```
Limit[binom[n, n / 2 - Log[n]/(2 Log[p]) + (1 + Log[2])/(2 Log[p])] *
  (n / 2 - Log[n]/(2 Log[p]) + (1 + Log[2])/(2 Log[p])) ^ (n / 2 + Log[n]/(2 Log[p]) - (1 + Log[2])/(2 Log[p])) *
  p^((n / 2 - Log[n]/(2 Log[p]) + (1 + Log[2])/(2 Log[p])) * (n / 2 + Log[n]/(2 Log[p]) - (1 + Log[2])/(2 Log[p]))) /
  ((2^(1/2 (1+n)) * n^(n/2 + Log[n]/(4 Log[p]) - (1+Log[2])/(2 Log[p]) - 1/2) * e^((1+Log[2])^2/(4 Log[p])) p^(n^2/4)) / Sqrt[π]), n → Infinity]

1
```

Now we find the limit

$$\lim_{n\to\infty}\frac{a_{rmax+m}}{a_{rmax}}=p^{-m^2}$$

```
Limit[binom[n, n/2 - Log[n]/(2 Log[p]) + (1 + Log[2])/(2 Log[p]) + m]/ binom[n, n/2 - Log[n]/(2 Log[p]) + (1 + Log[2])/(2 Log[p])],
 n → Infinity]
1
```

```
Limit[2^(n + 1/2)/Sqrt[Pi*n]*(n/2 - Log[n]/(2 Log[p]) + (1 + Log[2])/(2 Log[p]) + m)^(n/2 + Log[n]/(2 Log[p]) - (1 + Log[2])/(2 Log[p]) - m)*
  p^((n/2 - Log[n]/(2 Log[p]) + (1 + Log[2])/(2 Log[p]) + m)*(n/2 + Log[n]/(2 Log[p]) - (1 + Log[2])/(2 Log[p]) - m))/
  (2^(1/2 (1+n)) * n^(n/2 + Log[n]/(4 Log[p]) - (1+Log[2])/(2 Log[p]) - 1/2) * e^((1+Log[2])^2/(4 Log[p])) p^(n^2/4) / Sqrt[π]), n → Infinity]
p^-m^2
```

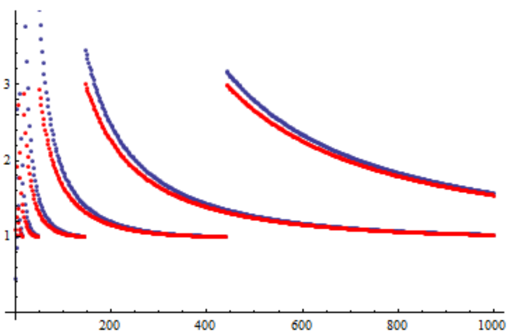
Value in the nearest integer point is then

$$a_{kmax}\ \sim\ a_{rmax}*p^{-(frac(rmax))^2}$$

or in case $frac(rmax)>1/2$

$$a_{kmax}\ \sim\ a_{rmax}*p^{-(1-frac(rmax))^2}$$

```
p = 3;
Show[
 ListPlot[
  ParallelTable[
   {n, (Binomial[n, n/2 - (Log[n/2] - 1)/(2 Log[p])] * (n/2 - (Log[n/2] - 1)/(2 Log[p]))^(n - (n/2 - (Log[n/2] - 1)/(2 Log[p]))) *
      p^((n/2 - (Log[n/2] - 1)/(2 Log[p])) * (n - (n/2 - (Log[n/2] - 1)/(2 Log[p])))))/
     (Binomial[n, Floor[n/2 - (Log[n/2] - 1)/(2 Log[p])]] * (Floor[n/2 - (Log[n/2] - 1)/(2 Log[p])])^(n - (Floor[n/2 - (Log[n/2] - 1)/(2 Log[p])])) *
      p^((Floor[n/2 - (Log[n/2] - 1)/(2 Log[p])]) * (n - (Floor[n/2 - (Log[n/2] - 1)/(2 Log[p])]))))}, {n, 1, 1000}]],
 ListPlot[Table[{n, p^(FractionalPart[n/2 - (Log[n/2] - 1)/(2 Log[p])])^2}, {n, 1, 1000}], PlotStyle → Red]]
```



Note that

$$\sum_{m=-\infty}^{\infty} p^{-(m-frac(rmax))^2} = \sum_{m=-\infty}^{\infty} p^{-(m-(1-frac(rmax)))^2}$$

Contributions of all terms are then

$$a_n = \sum_{k=0}^{n} p^{k(n-k)}\, k^{n-k} \binom{n}{k} \sim \sum_{m=-\infty}^{\infty} a_{kmax+m} \sim a_{rmax} * \sum_{m=-\infty}^{\infty} p^{-(m-frac(rmax))^2}$$

And final result is

$$\sum_{k=0}^{n} p^{k(n-k)}\, k^{n-k} \binom{n}{k} \sim \frac{1}{\sqrt{\pi}} * e^{\frac{(1+\log(2))^2}{4\log(p)}} * 2^{\frac{n+1}{2}} * n^{\frac{n-1}{2}+\frac{\log(n)}{4\log(p)}-\frac{1+\log(2)}{2\log(p)}} * p^{\frac{n^2}{4}} * \sum_{m=-\infty}^{\infty} p^{-\left(m-frac\left(\frac{n}{2}-\frac{\log\left(\frac{n}{2}\right)-1}{2\log(p)}\right)\right)^2}$$

Last term is not constant, but is **bounded**

$$\text{EllipticTheta}\left[2,0,\frac{1}{p}\right] = \sum_{m=-\infty}^{\infty} p^{-\left(m+\frac{1}{2}\right)^2} \le \sum_{m=-\infty}^{\infty} p^{-\left(m-frac\left(\frac{n}{2}-\frac{\log\left(\frac{n}{2}\right)-1}{2\log(p)}\right)\right)^2} \le \sum_{m=-\infty}^{\infty} p^{-m^2} = \text{EllipticTheta}\left[3,0,\frac{1}{p}\right]$$

```
Sum[p^-(m+1/2)^2, {m, -Infinity, Infinity}]
EllipticTheta[2, 0, 1/p]
```

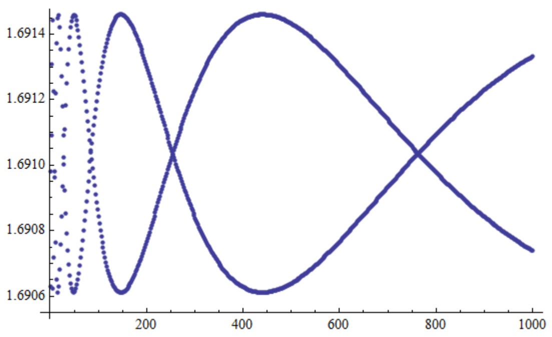
```
Sum[p^-m^2, {m, -Infinity, Infinity}]
EllipticTheta[3, 0, 1/p]
```

For example for $p = 3$ is

$$1.690611203075214233\ldots \le \sum_{m=-\infty}^{\infty} 3^{-\left(m-frac\left(\frac{n}{2}-\frac{\log\left(\frac{n}{2}\right)-1}{2\log(3)}\right)\right)^2} \le 1.691459681681715341\ldots$$

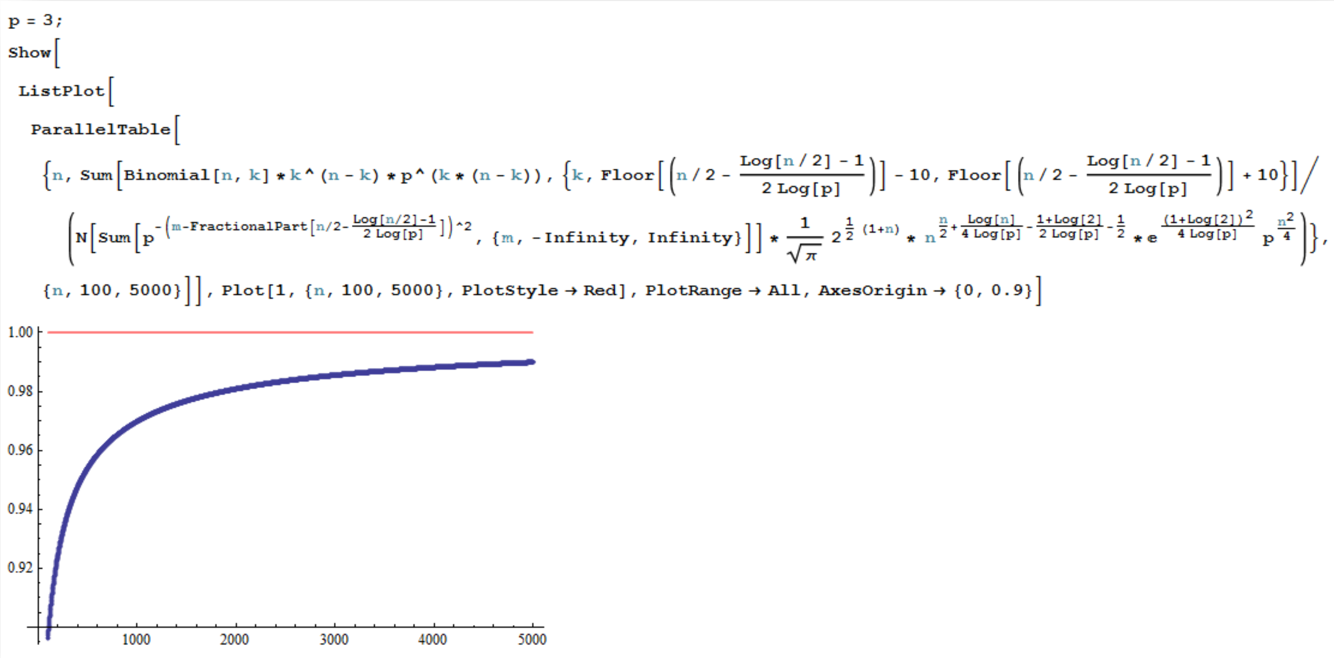
and here is graph

```
ListPlot[ParallelTable[N[Sum[p^-(m-FractionalPart[n/2-(Log[n/2]-1)/(2 Log[p])])^2, {m, -Infinity, Infinity}], 20],
  {n, 1, 1000}]]
```

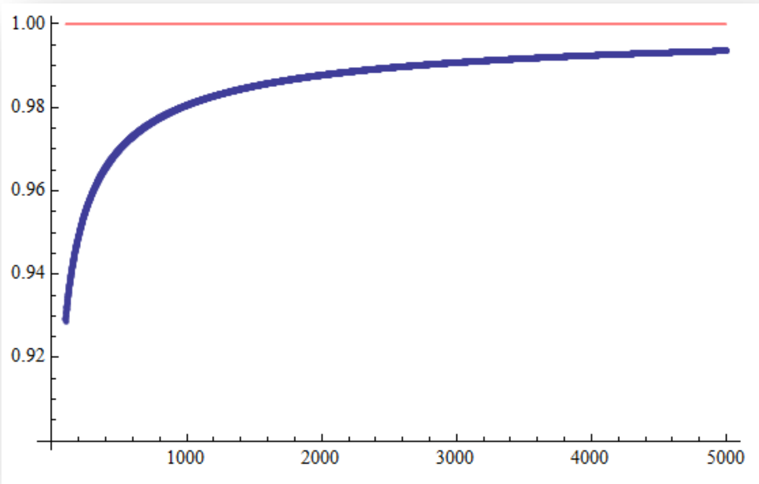


Numerical verification

$p = 3$



$p = 4$



References:

[1] OEIS - The On-Line Encyclopedia of Integer Sequences
[2] Kotěšovec V., Interesting asymptotic formulas for binomial sums, website 9.6.2013
[3] Kotěšovec V., Asymptotic of a sums of powers of binomial coefficients * x^k, website 20.9.2012

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