Asymptotic of subsequences of A212382

(Václav Kotěšovec, published July 17 2014)

In the OEIS (On-Line Encyclopedia of Integer Sequences) published Alois P. Heinz in 2012 sequences "Number of Dyck n-paths all of whose ascents have lengths equal to 1 (mod p)", which can be generalized (for $p \geq 1$) as the family of the sequences with an ordinary generating function $A(x)$, satisfies functional equation

$$A(x) = 1 + x \cdot \frac{A(x)}{1 - (x \cdot A(x))^p}$$

Sequences in the OEIS:
A000108 (p=1), A101785 (p=2), A212383 (p=3), A212384 (p=4), A212385 (p=5), A212386 (p=6), A212387 (p=7), A212388 (p=8), A212389 (p=9), A212390 (p=10), all sequences in one array together A212382.

**Theorem (V. Kotěšovec, July 16 2014):**
The asymptotic is (for $p \geq 1$)

$$a_n \sim \frac{s^2}{n^{3/2} r^{n-1/2} \sqrt{2\pi p (s - 1)} \left(1 + \frac{s}{1 + p (s - 1)}\right)}$$

where $r$ ($0 < r < 1$) and $s$ are real roots of the system of equations

$$r = \frac{p (s - 1)^2}{s (1 - p + ps)} \quad (rs)^p = \frac{s - 1 - rs}{s - 1}$$

**Proof:**

Following theorem by Edward A. Bender is (in case of implicit functions) very useful (for proof see [1], p.505 and also [4], p.469).

**Citation:** Edward A. Bender, "Asymptotic methods in enumeration" (1974), p.502, see [1]

**THEOREM 5.** Assume that the power series $w(z) = \sum a_n z^n$ with nonnegative coefficients satisfies $F(z, w) \equiv 0$. Suppose there exist real numbers $r > 0$ and $s > a_0$ such that

(i) for some $\delta > 0$, $F(z, w)$ is analytic whenever $|z| < r + \delta$ and $|w| < s + \delta$;
(ii) $F(r, s) = F_w(r, s) = 0$;
(iii) $F(r, s) \neq 0$, and $F_{w}(r, s) \neq 0$; and
(iv) if $|z| \leq r$, $|w| \leq s$, and $F(z, w) = F_w(z, w) = 0$, then $z = r$ and $w = s$.

Then

$$a_n \sim ((rF_r)/(2\pi F_{ww}))^{1/2} n^{-3/2} r^{-n}$$

where the partial derivatives $F_z$ and $F_{ww}$ are evaluated at $z = r$, $w = s$.

Bender's formula applied for ordinary generating function is

$$a_n \sim \frac{1}{n^{1/2} \sqrt{\frac{r F_z}{2\pi n F_{ww}}}}$$

(for exponential generating function see [8])
Now we have the implicit function

\[ f(x, y) = \frac{xy}{1 - (xy)^p} - y + 1 \]

<table>
<thead>
<tr>
<th>partial derivatives</th>
<th>( F_z )</th>
<th>( F_w )</th>
<th>( F_{ww} )</th>
</tr>
</thead>
</table>
| \( \frac{\partial}{\partial x} f(x, y) \) | \( \frac{y(p(xy)^p - (xy)^p + 1)}{(xy)^p - 1)^2} \) | \( \frac{x(p(xy)^p - (xy)^p + 1)}{(xy)^p - 1)^2} - 1 \) | \( \frac{-px(xy)^p(p(xy)^p - (xy)^p + p + 1)}{y((xy)^p - 1)^3} \)

r, s, are roots of the system of equations

\[ s \left( \frac{r}{1 - (rs)^p} - 1 \right) + 1 = 0 \quad \frac{r \left( p(rs)^p - (rs)^p + 1 \right)}{(rs)^p} = 1 \]

which can be simplified as

\[ r = \frac{p (s - 1)^2}{s (1 - p + ps)} \quad (rs)^p = \frac{s - 1 - rs}{s - 1} \]

Note that s is the root of a polynomial of degree 2p

\[ p^p (s - 1)^{2p} - (ps - p + 1)^{p-1} = 0 \]

The asymptotic is then

\[ a_n \sim \frac{1}{r^2 n^{-n}} \sqrt{-\frac{s^3 (rs)^{-p-1}(p(rs)^p - (rs)^p + 1)}{p(p(rs)^p - (rs)^p + p + 1)}} \]

after simplification

\[ a_n \sim \frac{s^2}{n^{3/2} r^{n-1/2} \sqrt{2\pi p (s - 1) \left( 1 + \frac{s}{1 + p (s - 1)} \right)}} \]
Numerical verification (for \( p=6 \), A212386)

```math
N[Solve[{r == p*(s-1)\^{2}/(s*(1-p*p*s)), (rs)^{p} == (s-1-res)/(s-1), r>0, r<1} /.
p -> 6, {r, s}, Reals], 20]
```

![Numerical verification graph for p=6](image)

Numerical verification (for \( p=9 \), A212389)

```math
N[Solve[{r == p*(s-1)\^{2}/(s*(1-p*p*s)), (rs)^{p} == (s-1-res)/(s-1), r>0, r<1} /.
p -> 9, {r, s}, Reals], 20]
```

![Numerical verification graph for p=9](image)

References:

[2] Kotšovec V., Asymptotic of implicit functions if \( F_{w} = 0 \), extension of theorem by Bender, website 19.1.2014

This article was published on the website [http://web.telecom.cz/vaclav.kotesovec/math.htm](http://web.telecom.cz/vaclav.kotesovec/math.htm), 17.7.2014 and in the [OEIS](http://oeis.org), July 17 2014