Asymptotic of the coefficients A079330 / A088989

(Václav Kotěšovec, published Aug 20 2014)

The positive solutions of the equation \( \tan(x) = x \) can be written explicitly in series form as

\[
x_k = \frac{\pi}{2} + \pi k - \sum_{n=1}^{\infty} \frac{d_{2n-1}}{(\pi/2 + \pi k)^{2n-1}}
\]

Coefficients can be found by series reversion of the series for \( x + \cot(x) \), (see [1] for more). Several first coefficients \( d_n \) are

\[
\text{CoefficientList[InverseSeries[Series[1/(\infty \cdot \cot[\infty]), \{x, 0, 16\}], x], x]}
\]

\[
\{0, 1, 0, \frac{2}{3}, 0, 0, 0, 0, \frac{13}{15}, 0, 0, 0, 0, 0, 0, 146, 0, 163, 0, 3465, 0, 631, 0, 675, 0, 3879594, 2027025}\}
\]

Numerators and denominators of the sequence \( d_{2n-1} \) are in the OEIS, see A079330 and A088989.

Main result:

\[
d_{2n-1} = \frac{A079330(n)}{A088989(n)} \sim \frac{\Gamma(1/3)}{2^{2/3} \cdot 3^{1/6} \cdot \pi^{5/3}} * \left(\frac{\pi}{7}\right)^{2n/3} / n^{4/3}
\]

where \( \Gamma \) is the Gamma function.

\[\text{Proof: We have an implicit function} \]

\[f(x, y) = y + \cot(y) - \frac{1}{x}\]

Second partial derivative \( \frac{\partial^2 f}{\partial y \partial y} \) at the point \( \left[\frac{2}{\pi}, \frac{\pi}{2}\right] \) is zero. In such case is not possible to apply theorem by Bender (see [3]), but asymptotic can be found using the Kotěšovec's extension of Bender's formula (see [2]).

<table>
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<th>notation in the theorem</th>
<th>partial derivatives</th>
<th>( r = 2/\pi, \quad s = \pi/2 )</th>
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</thead>
<tbody>
<tr>
<td>( F_x )</td>
<td>( \frac{\partial}{\partial x} f(x, y) )</td>
<td>( \frac{1}{x^2} )</td>
</tr>
<tr>
<td>( F_y )</td>
<td>( \frac{\partial}{\partial y} f(x, y) )</td>
<td>( 1 - \frac{1}{\sin^2(y)} )</td>
</tr>
<tr>
<td>( F_{yy} )</td>
<td>( \frac{\partial}{\partial y} \frac{\partial}{\partial y} f(x, y) )</td>
<td>( \frac{2 \cdot \cot(y)}{\sin^2(y)} )</td>
</tr>
<tr>
<td>( F_{yyy} )</td>
<td>( \frac{\partial}{\partial y} \frac{\partial}{\partial y} \frac{\partial}{\partial y} f(x, y) )</td>
<td>( -\frac{2}{\sin^4(y)} - \frac{4 \cdot \cot^2(y)}{\sin^2(y)} )</td>
</tr>
</tbody>
</table>

We have the system of equations:

\[f(r, s) = s + \cot(s) - \frac{1}{r} = 0 \quad f_y(r, s) = 1 - \frac{1}{\sin^2(s)} = 0\]

Roots are:

\[
r = \frac{1}{s} = \frac{1}{\pi} + k\pi \quad s = \frac{\pi}{2} + k\pi
\]

where \( k \) is integer.

Asymptotically is dominant only such root \( r \), whose absolute value is minimal.
We have two dominant solutions
\[ r = \frac{2}{\pi} \quad s = \frac{\pi}{2} \]
and
\[ r = -\frac{2}{\pi} \quad s = -\frac{\pi}{2} \]
where \( A_1 \) and \( A_2 \) are partial asymptotic.

From my theorem (see [2]) follows

\[
A_1 \sim \frac{1}{3} \frac{r^{1/3}}{F_{x\omega}} \left( \frac{\pi}{2} \right) n r^{-n} = \frac{r^{1/3}}{2\pi \sqrt{3}} \left( \frac{\pi}{2} \right) n r^{-n} = \frac{r^{1/3}}{2\pi \sqrt{3}} \left( \frac{\pi}{2} \right) n r^{-n} = \frac{r^{1/3}}{2\pi \sqrt{3}} \left( \frac{\pi}{2} \right) n r^{-n} = \frac{r^{1/3}}{2\pi \sqrt{3}} \left( \frac{\pi}{2} \right) n r^{-n} = \frac{r^{1/3}}{2\pi \sqrt{3}} \left( \frac{\pi}{2} \right) n r^{-n}
\]

\[ A_2 = -(-1)^n \cdot A_1 \]

Now
\[ d_n \sim 0 \quad \text{if } n \text{ is even} \]
and
\[ d_n \sim 2 A_1 \quad \text{if } n \text{ is odd} \]

After reindexing (odd terms only)
\[ n \rightarrow 2n - 1 \]

\[ d_{2n-1} \sim 2 \cdot \frac{r^{1/3}}{2\pi \sqrt{3}} \left( \frac{\pi}{2} \right) n^{2n-1} \sim \frac{r^{1/3}}{2\pi \sqrt{3}} \left( \frac{\pi}{2} \right) n^{2n-1} \]

Numerical verification (ratio tends to 1):

![Numerical verification graph](image)
References:

[2] V. Kotěšovec, Asymptotic of implicit functions if Fww = 0, extension of theorem by Bender, website 19.1.2014


Sequences A079330 and A088989, first root $x_1$ A115365

Related books and articles:
Du Bois Reymond’s Constants: A062546 (c2), A224196 (c3), A207528 (c4), A243108 (c5), A245333 (c6).

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