Asymptotic of the coefficients A079330 / A088989

(*Václav Kotěšovec, published Aug 20 2014*)

The positive solutions of the equation $tan(x) = x$ can be written explicitly in series form as

$$x_k = \frac{\pi}{2} + \pi k - \sum_{n=1}^{\infty} \frac{d_{2n-1}}{\left(\frac{\pi}{2} + \pi k\right)^{2n-1}}$$

Coefficients can be found by series reversion of the series for $x + cot(x)$, (see [1] for more).
Several first coefficients d_n are

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CoefficientList[InverseSeries[Series[1 / (x + Cot[x]), {x, 0, 16}], x], x]
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$$\left\{0,\ 1,\ 0,\ \frac{2}{3},\ 0,\ \frac{13}{15},\ 0,\ \frac{146}{105},\ 0,\ \frac{781}{315},\ 0,\ \frac{16328}{3465},\ 0,\ \frac{6316012}{675675},\ 0,\ \frac{38759594}{2027025}\right\}$$

Numerators and denominators of the sequence d_{2n-1} are in the OEIS, see A079330 and A088989.

Main result:

$$d_{2n-1} = \frac{A079330(n)}{A088989(n)} \sim \frac{\Gamma(1/3)}{2^{2/3}\ 3^{1/6}\ \pi^{5/3}} * \frac{\left(\frac{\pi}{2}\right)^{2n}}{n^{4/3}}$$

where Γ is the Gamma function

Proof: We have an implicit function

$$f(x,y) = y + \cot(y) - \frac{1}{x}$$

Second partial derivative $\frac{\partial}{\partial y}\frac{\partial}{\partial y} f(x,y)$ at the point $[\frac{2}{\pi}, \frac{\pi}{2}]$ is zero. In such case is not possible to apply theorem by Bender (see [3]), but asymptotic can be found using the Kotěšovec's extension of Bender's formula (see [2]).

notation in the theorem	partial derivatives		$r = 2/\pi,\ s = \pi/2$
F_z	$\frac{\partial}{\partial x} f(x,y)$	$\frac{1}{x^2}$	$f_x(r,s) = \frac{\pi^2}{4}$
F_w	$\frac{\partial}{\partial y} f(x,y)$	$1 - \frac{1}{\sin^2(y)}$	$f_y(r,s) = 0$
F_{ww}	$\frac{\partial}{\partial y}\frac{\partial}{\partial y} f(x,y)$	$\frac{2 * \cot(y)}{\sin^2(y)}$	$f_{yy}(r,s) = \mathbf{0}$
F_{www}	$\frac{\partial}{\partial y}\frac{\partial}{\partial y}\frac{\partial}{\partial y} f(x,y)$	$-\frac{2}{\sin^4(y)} - \frac{4 * \cot^2(y)}{\sin^2(y)}$	$f_{yyy}(r,s) = -2$

We have the system of equations:

$$f(r,s) = s + \cot(s) - \frac{1}{r} = 0 \qquad f_y(r,s) = 1 - \frac{1}{\sin^2(s)} = 0$$

Roots are:

$$r = \frac{1}{s} = \frac{1}{\frac{\pi}{2} + k\pi} \qquad s = \frac{\pi}{2} + k\pi$$

where *k* is integer.

Asymptotically is dominant only such root *r*, whose absolute value is minimal.

We have two dominant solutions

$$r = \frac{2}{\pi} \quad s = \frac{\pi}{2}$$

and

$$r = -\frac{2}{\pi} \quad s = -\frac{\pi}{2}$$

$$d_n \sim A_1 + A_2$$

where A_1 and A_2 are partial asymptotic.

From my theorem (see [2]) follows

$$A_1 \sim \frac{1}{3\,\Gamma\left(\frac{2}{3}\right) n\, r^n} \left(-\frac{6\, r\, F_z}{n\, F_{www}}\right)^{1/3} = \frac{\Gamma\left(\frac{1}{3}\right)}{2\pi\sqrt{3}\, n\, r^n} \left(-\frac{6\, r\, f_x(r,s)}{n\, f_{yyy}(r,s)}\right)^{1/3} = \frac{\Gamma\left(\frac{1}{3}\right)}{2\pi\sqrt{3}\, n \left(\frac{2}{\pi}\right)^{n-1/3}} \left(\frac{3\,\pi^2}{4\, n}\right)^{1/3} = \frac{\Gamma\left(\frac{1}{3}\right)}{2^{4/3}\, \pi^{2/3}\, 3^{1/6}\, n^{4/3}} \left(\frac{\pi}{2}\right)^n$$

$$A_2 = -(-1)^n * A_1$$

Now

$$d_n \sim 0 \quad \text{if } n \text{ is even}$$

and

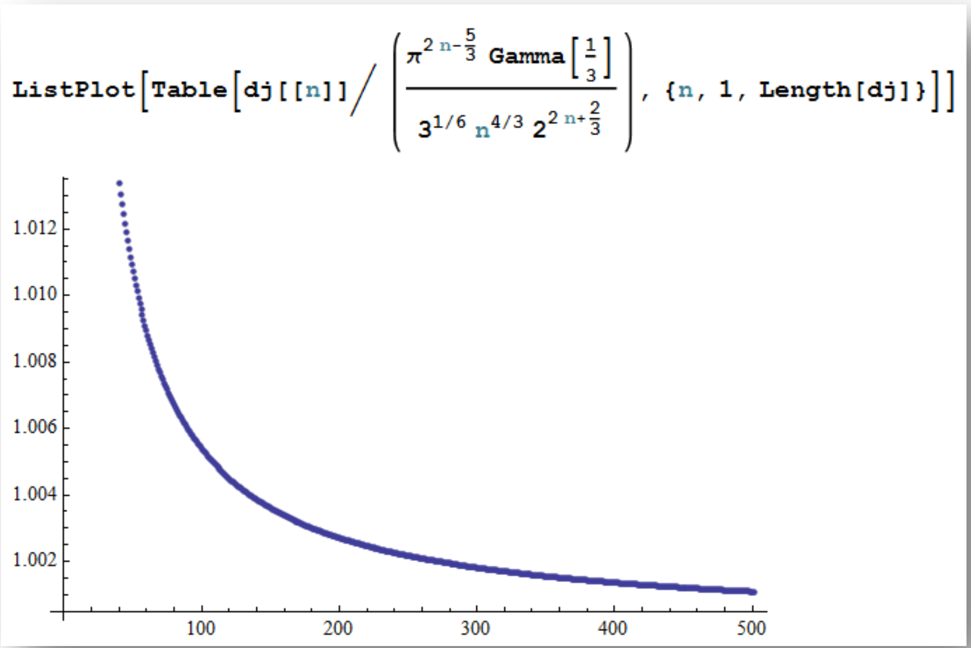
$$d_n \sim 2\, A_1 \quad \text{if } n \text{ is odd}$$

After reindexing (odd terms only)

$$n \to 2n - 1$$

$$d_{2n-1} \sim 2 * \frac{\Gamma\left(\frac{1}{3}\right)}{2^{4/3}\, \pi^{2/3}\, 3^{1/6}\, (2n-1)^{4/3}} * \left(\frac{\pi}{2}\right)^{2n-1} \sim \frac{\Gamma\left(\frac{1}{3}\right)}{2^{1/3}\, \pi^{2/3}\, 3^{1/6}\, (2n)^{4/3}} * \left(\frac{\pi}{2}\right)^{2n-1} = \frac{\Gamma\left(\frac{1}{3}\right)}{2^{2/3}\, \pi^{5/3}\, 3^{1/6}\, n^{4/3}} * \left(\frac{\pi}{2}\right)^{2n}$$

Numerical verification (ratio tends to 1):



References:

[1] Weisstein, Eric W. (and D. W. Cantrell), Tanc Function, MathWorld

[2] V. Kotěšovec, Asymptotic of implicit functions if Fww = 0, extension of theorem by Bender, website 19.1.2014

Theorem (V. Kotěšovec, 2013), see [3]
With same notation and same conditions as in theorem by Bender (see below), but if $F_{ww} = 0$ and $F_{www} \neq 0$ then

$$a_n \sim \frac{1}{3\,\Gamma\left(\frac{2}{3}\right)\,n\,r^n} * \left(-\frac{6\,r\,F_z}{n\,F_{www}}\right)^{1/3}$$

where Γ is the Gamma function
r is the radius of convergence
F_z is partial derivative of the function $F(z,w)$ at the point $[r,s]$
F_{www} is third partial derivative of the function $F(z,w)$ at the point $[r,s]$

[3] Edward A. Bender, "Asymptotic methods in enumeration" (1974), p.502

THEOREM 5. *Assume that the power series $w(z) = \sum a_n z^n$ with nonnegative coefficients satisfies $F(z, w) \equiv 0$. Suppose there exist real numbers $r > 0$ and $s > a_0$ such that*

(i) *for some $\delta > 0$, $F(z, w)$ is analytic whenever $|z| < r + \delta$ and $|q| < s + \delta$;*
(ii) $F(r, s) = F_w(r, s) = 0$;
(iii) $F_z(r, s) \neq 0$, *and* $F_{ww}(r, s) \neq 0$: *and*
(iv) *if $|z| \leqq r, |w| \leqq s$, and $F(z, w) = F_w(z, w) = 0$, then $z = r$ and $w = s$.*

Then

$$a_n \sim ((rF_z)/(2\pi F_{ww}))^{1/2} n^{-3/2} r^{-n}, \tag{7.1}$$

where the partial derivatives F_z and F_{ww} are evaluated at $z = r$, $w = s$.

[4] OEIS - The On-Line Encyclopedia of Integer Sequences
Sequences A079330 and A088989, first root x_1 A115365

Related books and articles:

[5] Watson, G. N., "Treatise on the Theory of Bessel Functions", 2nd ed., p. 502, 1922
[6] Young, R. M., "A Rayleigh Popular Problem", American Mathematical Monthly 93, pp. 660-664, 1986
[7] Watson, G. N., "Du Bois Reymond's Constants", Quarterly Journal of Mathematics 4, pp. 140-146, 1933
[8] Weisstein, Eric W., Du Bois Reymond Constants, MathWorld
[9] Finch, S. R. "Du Bois Reymond's Constants", Mathematical Constants, 3.12, pp. 237-240, 2003

Du Bois Reymond's Constants: A062546 (c2), A224196 (c3), A207528 (c4), A243108 (c5), A245333 (c6).

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