Asymptotic of generalized Apéry sequences with powers of binomial coefficients

(*Václav Kotěšovec, published 4.11.2012, extended 23.11.2012*)

Main result:

For $p > 0, q \geq 0, n \to \infty$ *is*

$$\sum_{k=0}^{n} \binom{n}{k}^p \binom{n+k}{k}^q \sim \frac{(1+r)^{qn}}{(1-r)^{pn+p}} * \sqrt{\frac{r\,(1-r^2)}{(p+q+(p-q)\,r) * (2\pi n)^{p+q-1}}}$$

where r is positive real root of the equation

$$(1-r)^p * (1+r)^q = r^{p+q}$$

Especially for $p = q > 0$

$$\sum_{k=0}^{n} \binom{n}{k}^p \binom{n+k}{k}^p \sim \frac{\left(1+\sqrt{2}\right)^{p\,(2n+1)}}{2^{p/2+3/4} * (\pi n)^{p-1/2} * \sqrt{p}}$$

and for $p = 2q > 0$

$$\sum_{k=0}^{n} \binom{n}{k}^{2q} \binom{n+k}{k}^q \sim \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{q\,(5n+4)-3/2}}{5^{1/4} * \sqrt{q\,(2\pi n)^{3q-1}}}$$

Partial results are already know, for $p = q = 2$ see [1] and [2], but always without closed form of the multiplicative constant. Case $p = q = 1$ see [10] or OEIS A001850.

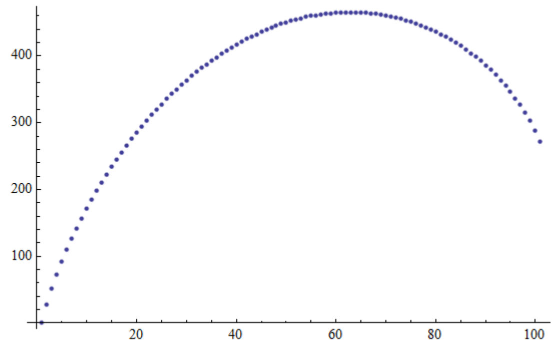
Proof of the general asymptotic formula

For general case we find (with using of Stirling formula and same method as in [3]) maximal term in the sum

$$\sum_{k=0}^{n} \binom{n}{k}^p \binom{n+k}{k}^q$$

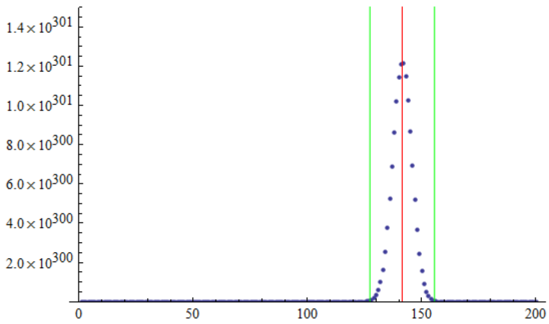
For first orientation see following graph (in logarithmical scale) for p = 4 and q = 2.

```
ListPlot[Table[Log[Binomial[n,k]^p*Binomial[n+k,k]^q]/.{p->4,q->2,n->100},{k,0,100}]]
```



Following graph is in normal scale for classic Apéry sequence, p = 2, q = 2

```
ListPlot[Table[Binomial[n,k]^p*Binomial[n+k,k]^q/.{p->2,q->2,n->200},{k,0,200}],PlotRange->All]
```



For finding of the maximum we must solve a equation

$$\frac{\mathrm{d}}{\mathrm{dk}}\binom{n}{k}^p\binom{n+k}{k}^q = 0$$

With help of program Mathematica:

```
stirling[n_]:=n^n/E^n*Sqrt[2*Pi*n];
binom[n_,k_]:=stirling[n]/stirling[k]/stirling[n-k];
Simplify[D[Simplify[binom[n,k]^p*binom[n+k,k]^q],k]]
```

$$-\frac{1}{k\ (k-n)\ (k+n)}\ 2^{\frac{1}{2}(-2-p-q)} \left(k^{-\frac{1}{2}-k}\ n^{\frac{1}{2}+n}\ (-k+n)^{-\frac{1}{2}+k-n}\right)^p \left(k^{-\frac{1}{2}-k}\ n^{-\frac{1}{2}-n}\ (k+n)^{\frac{1}{2}+k+n}\right)^q \pi^{\frac{1}{2}(-p-q)} \left(2 k^2 p + k n p - n^2 p + k n q - n^2 q + 2 k \left(k^2-n^2\right)(p+q)\, \mathrm{Log}[k] + 2 k \left(-k^2+n^2\right) p\, \mathrm{Log}[-k+n] - 2 k^3 q\, \mathrm{Log}[k+n] + 2 k n^2 q\, \mathrm{Log}[k+n]\right)$$

```
Simplify[(2 k^2 p + k n p - n^2 p + k n q - n^2 q + 2 k (k^2 - n^2) (p + q) Log[k] + 2 k (-k^2 +
n^2) p Log[-k + n] - 2 k^3 q Log[k + n] + 2 k n^2 q Log[k + n]) /. k -> (r*n)]
```

$$n^2 \left(-p-q+p r+q r+2 p r^2+2 n (p+q)\, r \left(-1+r^2\right) \mathrm{Log}[n\, r] + 2 n q r\, \mathrm{Log}[n\,(1+r)] - 2 n q r^3\, \mathrm{Log}[n\,(1+r)] - 2 n p r \left(-1+r^2\right) \mathrm{Log}[n-n\, r]\right)$$

Now we reduce terms which tends to zero if $n \to \infty$

```
Limit[((-p - q + p r + q r + 2 p r^2 + 2 n (p + q) r (-1 + r^2) Log[n r] + 2 n q r Log[n (1 + r)]
- 2 n q r^3 Log[n (1 + r)] - 2 n p r (-1 + r^2) Log[n - n r]))/n, n -> Infinity]
```

$$-2 r \left(-1+r^2\right) \left(p\, \mathrm{Log}[1-r] - (p+q)\, \mathrm{Log}[r] + q\, \mathrm{Log}[1+r]\right)$$

Maximum is in

$$k_{max} = r * n$$

where r is positive real root of the equation

$$(1-r)^p * (1+r)^q = r^{p+q}$$

Explicit form of the solution of this algebraic equation is simple in two cases:

$p = q$

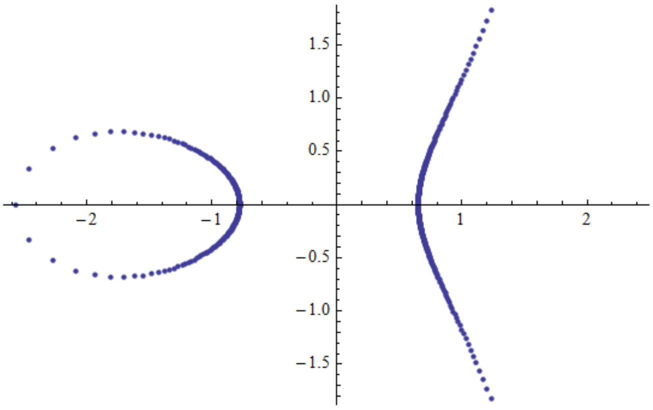
$$r = \frac{1}{\sqrt{2}}$$

$p = 2q$

$$r = \frac{\sqrt{5}-1}{2}$$

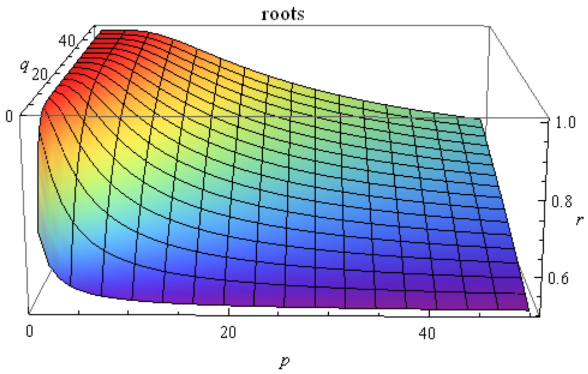
In general case is interesting distribution of all roots in the complex plane, for example if p=300 and q=200

```
sol = NSolve[(1 - r)^300 * (1 + r)^200 == r^500, 50];
rr = Table[r /. sol[[j]], {j, 1, Length[sol]}];
ListPlot[Table[{Re[rr[[j]]], Im[rr[[j]]]}, {j, 1, Length[rr]}]]
```



Positive real roots r for all p,q<=50 in 3D

```
pqmax = 50;
rrr = Flatten[Table[{p, q, sol = NSolve[(1 - r)^p * (1 + r)^q == r^(p + q), 20];
    Select[Table[r /. sol[[j]], {j, 1, Length[sol]}],
      Re[#] > 0 && Im[#] == 0 &][[1]]}, {p, 1, pqmax}, {q, 1, pqmax}]];
ListPlot3D[Table[{rrr[[3 * j - 2]], rrr[[3 * j - 1]], rrr[[3 * j]]},
  {j, 1, pqmax^2}], PlotRange → All, AxesLabel → {p, q, r},
 PlotLabel → roots, ColorFunction → "Rainbow"]
```



We will now analyze neighbourhood of maximum, $r * n \pm k$.
From logarithmic form of the Stirling formula we obtain (with help of Mathematica)

```
lognfak[n_]:=n*Log[n]-n+1/2*Log[n]+1/2*Log[2*Pi];
FullSimplify[(p-q)*lognfak[n]+q*lognfak[n+r*n+k]-(p+q)*lognfak[r*n+k]-p*lognfak[n-r*n-k]]
```

```
1/2 ((p - q) (2 n (-1 + Log[n]) + Log[2 n π]) -
    p (Log[-2 π (k + n (-1 + r))] - 2 (k + n (-1 + r)) (-1 + Log[-k + n - n r])) -
    (p + q) (-2 (k + n r) + Log[2] + Log[π] + (1 + 2 k + 2 n r) Log[k + n r]) +
    q (-2 (k + n + n r) + Log[2] + Log[π] + (1 + 2 k + 2 n (1 + r)) Log[k + n + n r]))
```

We now apply the first two terms from Taylor series (near 0)

$$\log(1+z) = z - \frac{z^2}{2} + \cdots$$

and approximate

```
slog[k_,n_]:=Log[n]+k/n-1/2*(k/n)^2;
```

```
FullSimplify[
 1/2 ((p - q) (2 n (-1 + Log[n]) + Log[2 n π]) -
    p (Log[2 π] + 2 (k + n (-1 + r)) + (1 - 2 k + 2 n (1 - r)) slog[-k, n - n r]) -
    (p + q) (-2 (k + n r) + Log[2] + Log[π] + (1 + 2 k + 2 n r) slog[k, n r]) +
    q (-2 (k + n + n r) + Log[2] + Log[π] + (1 + 2 k + 2 n (1 + r)) slog[k, n + n r]))]
```

$$\frac{1}{2}\Bigg((p-q)\,(2n(-1+\mathrm{Log}[n])+\mathrm{Log}[2n\pi]) - (p+q)\left(-2(k+nr)+\mathrm{Log}[2]+\mathrm{Log}[\pi]+(1+2k+2nr)\left(-\frac{k(k-2nr)}{2n^2r^2}+\mathrm{Log}[nr]\right)\right) + q\left(-2(k+n+nr)+\mathrm{Log}[2]+\mathrm{Log}[\pi]+(1+2k+2n(1+r))\left(-\frac{k(k-2n(1+r))}{2n^2(1+r)^2}+\mathrm{Log}[n(1+r)]\right)\right) - p\left(2(k+n(-1+r))+\mathrm{Log}[2]+\mathrm{Log}[\pi]+(1-2k-2n(-1+r))\left(-\frac{k(k-2n(-1+r))}{2n^2(-1+r)^2}+\mathrm{Log}[n-nr]\right)\right)\Bigg)$$

We now split this result to **three parts**, each we must return back to exponential form.

1) First part is independent on k

```
FullSimplify[
 Exp[-p Log[n π] + 1/2 p Log[2 n π] - 1/2 q Log[2 n π] - 1/2 p Log[4 - 4 r] + n p r Log[2 - 2 r] -
    n p Log[1 - r] - 1/2 p Log[r] - 1/2 q Log[r] - n q r Log[r] - n p r Log[2 r] + 1/2 q Log[1 + r] +
    n q Log[1 + r] + n q r Log[1 + r]]]
```

$$n^{\frac{1}{2}(-p-q)}\,(2\pi)^{\frac{1}{2}(-p-q)}\,(1-r)^{-p\left(\frac{1}{2}+n-nr\right)}\,r^{-\frac{1}{2}(p+q)(1+2nr)}\,(1+r)^{q\left(\frac{1}{2}+n+nr\right)}$$

But

$$(1-r)^p * (1+r)^q = r^{p+q}$$

and

$$\frac{(1+r)^{q*\left(\frac{1}{2}+n+nr\right)}}{(2\pi n)^{\frac{p+q}{2}} * (1-r)^{p*\left(\frac{1}{2}+n-nr\right)}\, r^{\frac{1}{2}*(p+q)*(1+2nr)}} = \frac{(1+r)^{q*\left(\frac{1}{2}+n\right)}}{(2\pi n)^{\frac{p+q}{2}} * (1-r)^{p*\left(\frac{1}{2}+n\right)}\, r^{\frac{1}{2}*(p+q)}} = \frac{(1+r)^{q*n}}{(2\pi n)^{\frac{p+q}{2}} * (1-r)^{p*(1+n)}}$$

2) Second part

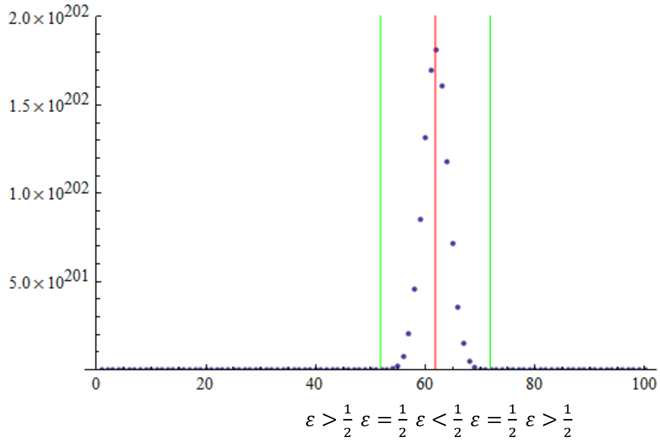
```
Simplify[k p Log[2 - 2 r] - k q Log[r] - k p Log[2 r] + k q Log[1 + r]]
k (p Log[1 - r] - (p + q) Log[r] + q Log[1 + r])
```

is dependent on k, but fortunately term is identical with equation for root r and is **equal to zero**!

3) Third part is interesting. Here are 3 cases. If we compare $k = n^{\varepsilon}$, then

if	limit	Exp(limit)
$\varepsilon < 1/2$	0	1
$\varepsilon = 1/2$	$\neq 0$	>1
$\varepsilon > 1/2$	$-\infty$	0

Graph for p=4, q=2, n=100



Green lines are $k = r * n - c * n^{\frac{1}{2}}$ and $k = r * n + c * n^{\frac{1}{2}}$, where $r * n$ is maximum and c is some constant. Therefore only case $\varepsilon \leq 1/2$ has some asymptotic weight (only between green lines are dominant terms). Values out of these bounds tends (in comparing with value in the maximum) asymptotically to zero.

Now we compute contributions of other terms (near maximum) yet

```
Limit[
 1/2 (-(p + q) ((1 + 2 k + 2 n r) (-(k (k - 2 n r))/(2 n^2 r^2))) + q ((1 + 2 k + 2 n (1 + r)) (-(k (k - 2 n (1 + r)))/(2 n^2 (1 + r)^2))) -
   p ((1 - 2 k - 2 n (-1 + r)) (-(k (k - 2 n (-1 + r)))/(2 n^2 (-1 + r)^2)))) /. k → (c * Sqrt[n]), n → Infinity]
```

$$\frac{c^2\,(p+q+p\,r-q\,r)}{2\,r\,(-1+r^2)}$$

$$c^2 = \frac{k^2}{n}$$

By merging of all parts we obtain

$$\sum_{k=0}^{n} \binom{n}{k}^{p} \binom{n+k}{k}^{q} \sim \frac{(1+r)^{q*n}}{(2\pi n)^{\frac{p+q}{2}} * (1-r)^{p*(1+n)}} * \sum_{k} exp\left(-\frac{k^2}{n} * \frac{(p+q+pr-qr)}{2r(1-r^2)}\right)$$

But

$$\sum_k e^{-\frac{k^2}{N}} \sim \sqrt{\pi N}$$

(for proof see [4], pp.482-485). Here is

$$N = \frac{2nr(1-r^2)}{p+q+pr-qr}$$

and final asymptotic expansion is

$$\sum_{k=0}^{n} \binom{n}{k}^p \binom{n+k}{k}^q \sim \frac{(1+r)^{qn}}{(1-r)^{pn+p}} * \sqrt{\frac{r\,(1-r^2)}{(p+q+(p-q)\,r) * (2\pi n)^{p+q-1}}}$$

or

$$\sum_{k=0}^{n} \binom{n}{k}^p \binom{n+k}{k}^q \sim \frac{(1+r)^{(2n+1)q}}{r^{(n+1)(p+q)}} * \sqrt{\frac{r\,(1-r^2)}{(p+q+(p-q)\,r) * (2\pi n)^{p+q-1}}}$$

In Mathematica format:

```
Sqrt[(r*(1-r^2))/((p+q+(p-q)*r)*(2*Pi*n)^(p+q-1))] * (1+r)^(q*n)/(1-r)^(p*n+p)
```

In special cases I found also detailed asymptotic:
if $p = q > 0$

$$\sum_{k=0}^{n} \binom{n}{k}^p \binom{n+k}{k}^p \sim \frac{\left(1+\sqrt{2}\right)^{p\,(2n+1)}}{2^{p/2+3/4} * (\pi n)^{p-1/2} * \sqrt{p}} * \left(1 - \frac{2p-1}{4n} + \frac{(4p^2+24p-19) * \sqrt{2}}{96\,pn}\right)$$

```
(1+Sqrt[2])^(p*(2*n+1))/(2^(p/2+3/4)*(Pi*n)^(p-1/2)*Sqrt[p])*
(1-(2*p-1)/(4*n)+(4*p^2+24*p-19)*Sqrt[2]/(96*p*n))
```

if $p = 2q > 0$

$$\sum_{k=0}^{n} \binom{n}{k}^{2q} \binom{n+k}{k}^q \sim \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{q\,(5n+4)\,-\,3/2}}{5^{1/4} * \sqrt{q\,(2\pi n)^{3q-1}}} * \left(1 - \frac{25q^2+96q-61}{120\,qn} - \frac{(13q^2-36q+17) * \sqrt{5}}{60\,qn}\right)$$

```
((1+Sqrt[5])/2)^(q*(5*n+4)-3/2)/(5^(1/4)*Sqrt[q*(2*Pi*n)^(3*q-1)])*
(1-(25*q^2+96*q-61)/(120*q*n)-(13*q^2-36*q+17)*Sqrt[5]/(60*q*n))
```

Numerical values of roots (relative positions of the maximum)

```
Do[Print[Table[sol=NSolve[(1-r)^p*(1+r)^q==r^(p+q),r,15];
Select[Table[r/.sol[[j]],{j,1,Length[sol]}],Re[#]>0&&Im[#]==0&][[1]],{q,1,6}]],{p,1,6}]
```

	q=1	q=2	q=3	q=4	q=5	q=6
p=1	0.707106781186548	0.829483540958497	0.903408192018871	0.946998934922572	0.971747074436518	0.985277465769138
p=2	0.618033988749895	0.707106781186548	0.775918595324391	0.829483540958497	0.871156755860518	0.903408192018871
p=3	0.582522207810478	0.650398476698676	0.707106781186548	0.754877666246693	0.795278035987325	0.829483540958497
p=4	0.563442462497052	0.618033988749895	0.665520266799288	0.707106781186548	0.743678979444120	0.775918595324391
p=5	0.551531326365234	0.597132308657428	0.637804575024936	0.674277548113842	0.707106781186548	0.736730665129491
p=6	0.543386978253671	0.582522207810478	0.618033988749895	0.650398476698676	0.679989062840487	0.707106781186548

Numerical verification of the general asymptotic formula

```
n=10000;Do[Print[Table[Clear[r];sol=NSolve[(1-r)^p*(1+r)^q==r^(p+q),r,50];
r=Select[Table[r/.sol[[j]],{j,1,Length[sol]}],Re[#]>0&&Im[#]==0&][[1]];
N[Sum[Binomial[n,k]^p*Binomial[n+k,k]^q,{k,0,n}]/
(Sqrt[(r*(1-r^2))/((p+q+(p-q)*r)*(2*Pi*n)^(p+q-1))]*(1+r)^(q*n)/(1-r)^(p*n+p)),10]
,{q,1,6}]],{p,1,6}]
```

quotient = Sum / asymptotic, for n=10000

	q=1	q=2	q=3	q=4	q=5	q=6
p=1	0.9999882584	0.9999754589	0.9999626873	0.9999500438	0.9999374955	0.9999249900
p=2	0.9999723610	0.9999581465	0.9999451757	0.9999340121	0.9999255429	0.9999210665
p=3	0.9999499757	0.9999339847	0.9999187059	0.9999042815	0.9998909672	0.9998791330
p=4	0.9999262866	0.9999093438	0.9998928818	0.9998769348	0.9998615946	0.9998470042
p=5	0.9999020891	0.9998845788	0.9998674263	0.9998506324	0.9998342326	0.9998182912
p=6	0.9998776355	0.9998597553	0.9998421574	0.9998248317	0.9998077908	0.9997910659

p=1, q=1, A001850

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k} \sim \frac{\left(1+\sqrt{2}\right)^{2n+1}}{2^{5/4} \sqrt{\pi n}}$$

p=2, q=2, A005259
(Apéry sequence)

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^2 \sim \frac{\left(1+\sqrt{2}\right)^{4n+2}}{2^{9/4}\, \pi^{3/2}\, n^{3/2}}$$

p=3, q=3, A092813

$$\sum_{k=0}^{n} \binom{n}{k}^3 \binom{n+k}{k}^3 \sim \frac{\left(1+\sqrt{2}\right)^{3\,(2n+1)}}{2^{9/4} * (\pi n)^{5/2} * \sqrt{3}}$$

p=4, q=4, A092814

$$\sum_{k=0}^{n} \binom{n}{k}^4 \binom{n+k}{k}^4 \sim \frac{\left(1+\sqrt{2}\right)^{4\,(2n+1)}}{2^{15/4} * (\pi n)^{7/2}}$$

p=5, q=5, A092815

$$\sum_{k=0}^{n} \binom{n}{k}^5 \binom{n+k}{k}^5 \sim \frac{\left(1+\sqrt{2}\right)^{5\,(2n+1)}}{2^{13/4} * (\pi n)^{9/2} * \sqrt{5}}$$

p=6, q=6, A218689

$$\sum_{k=0}^{n} \binom{n}{k}^6 \binom{n+k}{k}^6 \sim \frac{\left(1+\sqrt{2}\right)^{6\,(2n+1)}}{2^{17/4} * (\pi n)^{11/2} * \sqrt{3}}$$

p=2, q=1, A005258

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k} \sim \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{5n+5/2}}{2\pi n * 5^{1/4}} = \frac{\left(\frac{11+5\sqrt{5}}{2}\right)^{n+\frac{1}{2}}}{2\pi n * 5^{1/4}}$$

p=4, q=2, A218690

$$\sum_{k=0}^{n} \binom{n}{k}^4 \binom{n+k}{k}^2 \sim \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{2(5n+4)-3/2}}{5^{1/4} * (2\pi n)^{5/2} * \sqrt{2}}$$

p=6, q=3, A218692

$$\sum_{k=0}^{n} \binom{n}{k}^6 \binom{n+k}{k}^3 \sim \frac{\left(\frac{1+\sqrt{5}}{2}\right)^{3(5n+4)-3/2}}{5^{1/4} * (2\pi n)^4 * \sqrt{3}}$$

p=1, q=2, A112019

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k}^2 \sim \frac{(1+r)^{4n+5/2}}{r^{3n+5/2}} * \frac{1}{2\pi n} * \sqrt{\frac{1-r}{3-r}}$$

where r is positive real root of the equation

$$(1-r)*(1+r)^2 = r^3$$

$$r = \frac{1}{6}\left(\left(44-3\sqrt{177}\right)^{1/3} + \left(44+3\sqrt{177}\right)^{1/3} - 1\right) = 0.829483540958497 \ldots$$

p=1, q=3, A218693

$$\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{k}^3 \sim \frac{(1+r)^{6n+7/2}}{r^{4n+7/2}} * \frac{1}{4\pi^{3/2}\, n^{3/2}} * \sqrt{\frac{1-r}{2-r}}$$

where r is positive real root of the equation

$$(1-r)*(1+r)^3 = r^4$$

$$r = 0.903408192018871 \ldots$$

p=2, q=3, A111968

$$\sum_{k=0}^{n} \binom{n}{k}^2 \binom{n+k}{k}^3 \sim \frac{(1+r)^{6n+7/2}}{r^{5n+9/2}} * \frac{1}{4\pi^2 n^2} * \sqrt{\frac{1-r}{5-r}}$$

where r is positive real root of the equation

$$(1-r)^2*(1+r)^3 = r^5$$

$$r = 0.775918595324391 \ldots$$

p=3, q=1, A014178

$$\sum_{k=0}^{n} \binom{n}{k}^3 \binom{n+k}{k} \sim \frac{(1+r)^{2n+3/2}}{r^{4n+7/2}} * \frac{1}{4\pi^{3/2}\, n^{3/2}} * \sqrt{\frac{1-r}{2+r}}$$

where r is positive real root of the equation

$$(1-r)^3*(1+r) = r^4$$

$$r = 0.582522207810478 \ldots$$

p=3, q=2, A014180

$$\sum_{k=0}^{n} \binom{n}{k}^3 \binom{n+k}{k}^2 \sim \frac{(1+r)^{4n+5/2}}{r^{5n+9/2}} * \frac{1}{4\pi^2 n^2} * \sqrt{\frac{1-r}{5+r}}$$

where r is positive real root of the equation

$$(1-r)^3*(1+r)^2 = r^5$$

$$r = 0.650398476698676 \ldots$$

Case p=0

(extension of this article, added 23.11.2012)

If $p = 0$ then maximum is at the point $k = n$. With same method as in previous section we obtain

$$\sum_{k=0}^{n} \binom{n+k}{k}^q \sim \binom{2n}{n}^q * \sum_{k=0}^{\infty} 2^{-kq} = \binom{2n}{n}^q * \frac{2^q}{2^q - 1}$$

and finally for $p = 0,\ q > 0, n \to \infty$ is

$$\sum_{k=0}^{n} \binom{n+k}{k}^q \sim \frac{2^{(2n+1)*q}}{(2^q - 1) * (\pi n)^{q/2}}$$

Detailed asymptotic is then

$$\sum_{k=0}^{n} \binom{n+k}{k}^q \sim \frac{2^{(2n+1)*q}}{(2^q - 1) * (\pi n)^{q/2}} * \left(1 - \frac{q}{2n} * \left(\frac{1}{4} + \frac{1}{(2^q - 1)^2}\right) + O\left(\frac{1}{n^2}\right)\right)$$

p=0, q=1, A001700

$$\sum_{k=0}^{n} \binom{n+k}{k} = \binom{2n+1}{n+1} \sim \frac{2^{2n+1}}{\sqrt{\pi n}}$$

p=0, q=2, A112029

$$\sum_{k=0}^{n} \binom{n+k}{k}^2 \sim \frac{2^{4n+2}}{3\pi n}$$

p=0, q=3, A112028

$$\sum_{k=0}^{n} \binom{n+k}{k}^3 \sim \frac{2^{6n+3}}{7 * (\pi n)^{3/2}}$$

p=0, q=4, A219562

$$\sum_{k=0}^{n} \binom{n+k}{k}^4 \sim \frac{2^{8n+4}}{15\pi^2 n^2}$$

p=0, q=5, A219563

$$\sum_{k=0}^{n} \binom{n+k}{k}^5 \sim \frac{2^{10n+5}}{31 * (\pi n)^{5/2}}$$

p=0, q=6, A219564

$$\sum_{k=0}^{n} \binom{n+k}{k}^6 \sim \frac{2^{12n+6}}{63\pi^3 n^3}$$

Sequences in OEIS

	q=0	q=1	q=2	q=3	q=4	q=5	q=6
p=0		A001700	A112029	A112028	A219562	A219563	A219564
p=1	A000079	A001850	A112019	A218693			
p=2	A000984	A005258	A005259	A111968			
p=3	A000172	A014178	A014180	A092813			
p=4	A005260		A218690		A092814		
p=5	A005261					A092815	
p=6	A069865			A218692			A218689

Recurrence order

	q=0	q=1	q=2	q=3	q=4	q=5	q=6
p=0	-	2	2	3	3	5	4
p=1	1	2	3	4	5	6	7
p=2	1	2	2	4	5	6	7
p=3	2	4	5	6	7	8	9
p=4	2	4	5	6	6	8	9
p=5	3	6	7	8	9	10	11
p=6	3	6	7	8	9	10	10

Maximal degree of polynomials in recurrence

	q=0	q=1	q=2	q=3	q=4	q=5	q=6
p=0	-	1	4	10	13	36	33
p=1	0	1	3	8	16	29	47
p=2	1	2	3	12	24	41	65
p=3	2	8	16	25	47	72	104
p=4	3	13	24	38	43	88	127
p=5	6	29	47	72	104	129	195
p=6	9	42	65	92	127	170	175

(for some recurrences see OEIS links)

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Czech summary:

Článek obsahuje odvození asymptotických vzorců pro obecnou skupinu sum součinů mocnin binomických koeficientů, z nichž speciálním případem (pro p=2 a q=2 nebo q=1) jsou tzv. Apéryho sekvence (které se jinak používají v důkazu, že Zeta(3) je iracionální). Pro tyto dvě speciální sekvence byly sice již asymptotické vzorce odvozeny, ale nikde nebyly uvedeny hodnoty multiplikativních konstant. Podařilo se mi tyto konstanty vyjádřit v symbolickém tvaru.
Metoda odvození obecných vzorců spočívá v nalezení maximálního členu v sumě a v určení pásu kolem tohoto maxima (řádu $\sqrt{n}$), který je ještě asymptoticky významný (v limitě proti maximu jsou pak příspěvky ostatních členů mimo tento pás už nulové). Odvodil jsem algebraickou rovnici, jejímž jediným kladným reálným kořenem je bod, ve kterém nastává maximum. Elegantně jej lze vyjádřit pro případy p=q a p=2q. Pointou důkazu je místo, kdy se část asymptotického rozvoje ukáže jako nulová právě s použitím rovnice pro tento kořen. Rovnice pro bod maxima je pak použita ještě jednou i k výraznému zjednodušení výsledného asymptotického vzorce.

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