Asymptotics of the sequence A120733

(*Václav Kotěšovec, published May 03 2015*)

The sequence A120733 in OEIS is "*Number of matrices with nonnegative integer entries and without zero rows or columns such that sum of all entries is equal to n*".

Main result:

$$A120733(n) \sim \frac{2^{\frac{\log(2)}{2}-2} * n!}{\left(\log(2)\right)^{2n+2}}$$

Proof:

In OEIS we have a formula

$$A120733(n) = \frac{1}{n!} * \sum_{k=1}^{n} (-1)^{n-k} * S_1(n,k) * A000670(k)^2$$

where $S_1(n,k)$ are the Stirling numbers of the first kind
and A000670 are Fubini numbers (number of ordered partitions of n, the ordered Bell numbers)

The sequence A000670 has an exponential generating function

$$f(x) = \frac{1}{2 - e^x}$$

with a simple pole at $r = \log(2)$ and the derivative is

$$f'(x) = \frac{e^x}{(2 - e^x)^2}$$

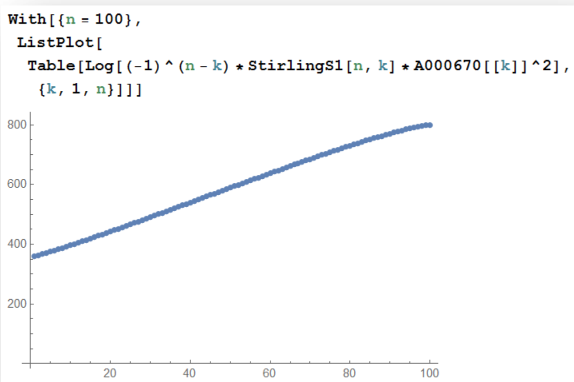
Asymptotic is then

$$A000670(n) \sim -\frac{\text{residue}(f,r)}{r^{n+1}} * n! = \frac{f(r)^2}{f'(r) * r^{n+1}} * n! = \frac{n!}{2 * (\log(2))^{n+1}}$$

Now

$$A120733(n) = \frac{1}{n!} * A000670(n)^2 * \sum_{k=1}^{n} (-1)^{n-k} * S_1(n,k) * \left(\frac{A000670(k)}{A000670(n)}\right)^2$$

The maximal term in the sum is at the position $k = n$ (see a graph in the logarithmical scale)



and the sum can be rewritten as

$$\sum_{k=1}^{n} (-1)^{n-k} * S_1(n,k) * \left(\frac{A000670(k)}{A000670(n)}\right)^2 = 1 - S_1(n,n-1) * \left(\frac{A000670(n-1)}{A000670(n)}\right)^2 + S_1(n,n-2) * \left(\frac{A000670(n-2)}{A000670(n)}\right)^2 - \cdots$$

For **fixed *k*** we have (see H. W. Gould, formula 8.4):

$$(-1)^k * S_1(n,n-k) \sim \frac{n^{2k}}{2^k\, k!}$$

$$\frac{A000670(n-k)}{A000670(n)} \sim \frac{(n-k)!}{n!} * (\log(2))^k \sim \left(\frac{\log(2)}{n}\right)^k$$

Together

$$(-1)^k * S_1(n, n-k) * \left(\frac{A000670(n-k)}{A000670(n)}\right)^2 \sim \frac{n^{2k}}{2^k\, k!} * \left(\frac{\log(2)}{n}\right)^{2k} = \frac{(\log(2))^{2k}}{2^k\, k!}$$

Contribution of all terms in the sum is

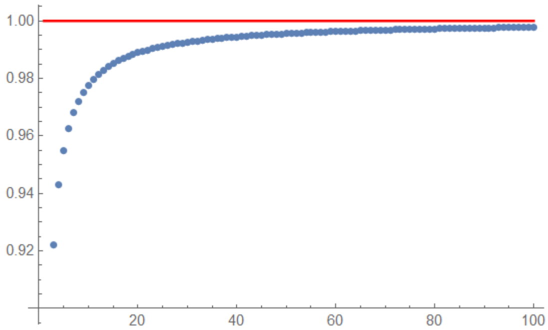
$$\sum_{k=0}^{\infty} \frac{(\log(2))^{2k}}{2^k\, k!} = 1 + \frac{(\log(2))^2}{2} + \frac{(\log(2))^4}{8} + \frac{(\log(2))^6}{48} + \frac{(\log(2))^8}{384} + \cdots = e^{\frac{(\log(2))^2}{2}}$$

The final asymptotic is

$$A120733(n) \sim \frac{1}{n!} * \left(\frac{n!}{2 * (\log(2))^{n+1}}\right)^2 * e^{\frac{(\log(2))^2}{2}} = \frac{2^{\frac{\log(2)}{2} - 2} * n!}{(\log(2))^{2n+2}}$$

Numerical verification, the ratio tends to 1**:**

```
Show[Plot[1, {n, 1, Length[A120733]}, PlotStyle → Red],
 ListPlot[Table[A120733[[n]] / (2^(Log[2] / 2 - 2) * n! / (Log[2]) ^ (2 * n + 2)),
   {n, 1, Length[A120733]}]], PlotRange → {0.9, 1}, AxesOrigin → {0, 0.9}]
```



Richardson extrapolation, 10 steps, from 100 terms of the sequence. The convergence is very good.

```
funs[n_] := A120733[[n]] / (2^(Log[2] / 2 - 2) * n! / (Log[2]) ^ (2 * n + 2));
Do[
 Print[
  N[Sum[(-1) ^ (m + j) * funs[j * Floor[Length[A120733] / m]] *
     j^(m - 1) / (j - 1) ! / (m - j) !, {j, 1, m}], 50]], {m, 1, 10}]
0.99778816833726945241661234815737320703196410564172
1.0000039574626725055762594996210106473825462922470
0.99999988055345632461574549530315345058030626984981
1.0000000042838525561154160317693983789664426252250
0.99999999979490020168053314867089076999656317118801
1.0000000000163891837102709581690901941273584305202
0.99999999999888049573829528543064821038426019693485
1.0000000000001270121098487290611284107787655712292
0.999999999999998910807403868822464799778437998030794
1.0000000000000012326832489317331949436542859743384
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