The integration of q-series

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1) The integrals of type

$$\int_0^1 \prod_{k=1}^{\infty} (1 - q^{mk})\, dq$$

(where $m > 0$, fixed)

$\boldsymbol{m = 1}$, OEIS A258232

$$\int_0^1 \prod_{k=1}^{\infty} (1 - q^{k})\, dq = \frac{8\,\pi\sqrt{\frac{3}{23}}\,\sinh\left(\frac{\sqrt{23}\,\pi}{6}\right)}{2\cosh\left(\frac{\sqrt{23}\,\pi}{3}\right) - 1} = 0.368412535931433652321316 59\,...$$

Proof: From the Euler pentagonal theorem [2], [3], [5], [6], [8], [9] follows

$$\int_0^1 \prod_{k=1}^{\infty} (1 - q^{k})\, dq = \int_0^1 \sum_{n=-\infty}^{\infty} (-1)^n\, q^{n\,(3n-1)/2}\, dq = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\frac{n\,(3n-1)}{2} + 1} = \frac{8\,\pi\sqrt{\frac{3}{23}}\,\sinh\left(\frac{\sqrt{23}\,\pi}{6}\right)}{2\cosh\left(\frac{\sqrt{23}\,\pi}{3}\right) - 1}$$

$\boldsymbol{m = 2}$, A258408

$$\int_0^1 \prod_{k=1}^{\infty} (1 - q^{2k})\, dq = \frac{4\,\pi\sqrt{\frac{3}{11}}\,\sinh\left(\frac{\sqrt{11}\,\pi}{6}\right)}{2\cosh\left(\frac{\sqrt{11}\,\pi}{3}\right) - 1} = 0.577332120183979705552546962\,...$$

In general

$$\int_0^1 \prod_{k=1}^{\infty} (1 - q^{mk})\, dq = \int_0^1 \sum_{n=-\infty}^{\infty} (-1)^n\, q^{mn\,(3n-1)/2}\, dq = \sum_{n=-\infty}^{\infty} \frac{(-1)^n}{\frac{mn(3n-1)}{2} + 1}$$

if $0 < m < 24$

$$\int_0^1 \prod_{k=1}^{\infty} (1 - q^{mk})\, dq = \frac{8\,\pi\,\sqrt{3}\,\sinh\left(\frac{\pi}{6}\sqrt{\frac{24}{m} - 1}\right)}{\left(2\cosh\left(\frac{\pi}{3}\sqrt{\frac{24}{m} - 1}\right) - 1\right)\sqrt{m\,(24 - m)}}$$

if $m = 24$ (A258414)

$$\int_0^1 \prod_{k=1}^{\infty} (1 - q^{24k})\, dq = \frac{\pi^2}{6\sqrt{3}} = 0.949703126294\,...$$

if $m > 24$

$$\int_0^1 \prod_{k=1}^{\infty} (1 - q^{mk})\, dq = \frac{8\,\pi\,\sqrt{3}\,\sin\left(\frac{\pi}{6}\sqrt{1 - \frac{24}{m}}\right)}{\left(2\cos\left(\frac{\pi}{3}\sqrt{1 - \frac{24}{m}}\right) - 1\right)\sqrt{m\,(m - 24)}}$$

The following integral (where $m > 0$) tends to infinity, the asymptotic growth is

$$\int_0^1 \prod_{k=1}^{n} (1 + q^{mk})\, dq \sim \frac{2\,(m+1)^{n+1}}{m\,n^2}$$

Special case for $m = 1$

$$\int_0^1 \prod_{k=1}^{n} (1 + q^{k})\, dq \sim \frac{2^{n+2}}{n^2}$$

2) The integrals of type

$$\int_0^1 \prod_{k=1}^{\infty} (1-q^k)^m \, dq$$

(where $m > 0$, fixed)

$\boldsymbol{m = 1}$, A258232 (again, for completeness)

$$\int_0^1 \prod_{k=1}^{\infty} (1-q^k) \, dq = \frac{8\pi\sqrt{\frac{3}{23}}\sinh\left(\frac{\sqrt{23}\,\pi}{6}\right)}{2\cosh\left(\frac{\sqrt{23}\,\pi}{3}\right)-1} = 0.36841253593143365232131659\ldots$$

$\boldsymbol{m = 2}$, A258406

$$\int_0^1 \prod_{k=1}^{\infty} (1-q^k)^2 \, dq = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{8\,(n+1)\,(-1)^n}{(n^2-2k^2+2kn+n+2)\,(n^2-2k^2+2kn+5n+6)} = 0.25387408237827600298850889\ldots$$

Proof: From the Hecke-Rogers identity [3] follows

$$\int_0^1 \prod_{k=1}^{\infty} (1-q^k)^2 \, dq = \int_0^1 \sum_{n=0}^{\infty} \sum_{j=-\left\lfloor\frac{n}{2}\right\rfloor}^{\left\lfloor\frac{n}{2}\right\rfloor} (-1)^{n+j} * q^{\frac{n\,(n+1)}{2}-\frac{j\,(3j-1)}{2}} \, dq = \sum_{n=0}^{\infty} \sum_{j=-\left\lfloor\frac{n}{2}\right\rfloor}^{\left\lfloor\frac{n}{2}\right\rfloor} \frac{(-1)^{n+j}}{\frac{n\,(n+1)}{2}-\frac{j\,(3j-1)}{2}+1}$$

From the similar formula by Andrews [4], [3] follows

$$\int_0^1 \prod_{k=1}^{\infty} (1-q^k)^2 \, dq = \int_0^1 \sum_{n=0}^{\infty}\sum_{k=0}^{n} (-1)^n q^{n\,(n+1)/2}\,(1-q^{2n+2})\,q^{k\,(n-k)} \, dq = \sum_{n=0}^{\infty}\sum_{k=0}^{n} \frac{8\,(n+1)\,(-1)^n}{(n^2-2k^2+2kn+n+2)\,(n^2-2k^2+2kn+5n+6)}$$

$\boldsymbol{m = 3}$, A258407

$$\int_0^1 \prod_{k=1}^{\infty} (1-q^k)^3 \, dq = \frac{2\,\pi}{\cosh\left(\frac{\pi\sqrt{7}}{2}\right)} = 0.19688061531458897535335135 8\ldots$$

Proof: From the Jacobi triple product [2], [3], [6], [7], [8], [9] follows

$$\int_0^1 \prod_{k=1}^{\infty} (1-q^k)^3 \, dq = \int_0^1 \sum_{n=0}^{\infty} (-1)^n\,(2n+1)\,q^{n\,(n+1)/2} \, dq = \sum_{n=0}^{\infty} \frac{(-1)^n\,(2n+1)}{\frac{n\,(n+1)}{2}+1} = \frac{2\,\pi}{\cosh\left(\frac{\pi\sqrt{7}}{2}\right)}$$

In general

$$\int_0^1 \prod_{k=1}^{\infty} \left(1-q^{jk}\right)^3 \, dq = \int_0^1 \sum_{n=0}^{\infty} (-1)^n\,(2n+1)\,q^{jn\,(n+1)/2} \, dq = \sum_{n=0}^{\infty} \frac{(-1)^n\,(2n+1)}{\frac{jn(n+1)}{2}+1}$$

if 0 < j < 8

$$\int_0^1 \prod_{k=1}^{\infty} \left(1-q^{jk}\right)^3 \, dq = \frac{2\pi}{j\cosh\left(\frac{\pi}{2}\sqrt{\frac{8}{j}-1}\right)}$$

if j = 8

$$\int_0^1 \prod_{k=1}^{\infty} (1-q^{8k})^3 \, dq = \frac{\pi}{4}$$

if j > 8

$$\int_0^1 \prod_{k=1}^{\infty} \left(1-q^{jk}\right)^3 \, dq = \frac{2\pi}{j\cos\left(\frac{\pi}{2}\sqrt{1-\frac{8}{j}}\right)}$$

Special values:

$$\int_0^1 \prod_{k=1}^{\infty} (1-q^{4k})^3 \, dq = \frac{\pi}{2\cosh\left(\frac{\pi}{2}\right)} = \frac{\pi}{e^{-\pi/2}+e^{\pi/2}} \qquad \int_0^1 \prod_{k=1}^{\infty} (1-q^{9k})^3 \, dq = \frac{4\pi}{9\sqrt{3}}$$

This efficient program under the Mathematica compute the integral numerically (for m=4, from "nmax" terms)

```
nmax=200; p=1;
q4=Table[PrintTemporary[n];
p=Expand[p*(1-x^n)^4]; Total[CoefficientList[p,x]/Range[1,Exponent[p,x]+1]],{n,1,nmax}];
q4n=N[q4,1000];
Table[SequenceLimit[Take[q4n,j]],{j,Length[q4n]-100,Length[q4n],10}]
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$\boldsymbol{m=4}$, A258404

$$\int_0^1 \prod_{k=1}^{\infty}(1-q^k)^4 \; dq = 0.16182024229485656180261334985786534 3\ldots$$

$\boldsymbol{m=5}$, A258405

$$\int_0^1 \prod_{k=1}^{\infty}(1-q^k)^5 \; dq = 0.13780107084655464284538613140219384 3\ldots$$

3) Miscellaneous integrals

A258412

$$\int_0^1 \prod_{k=1}^{\infty}(1-q^k)^k \; dq = 0.2987833651065672987709537721 14\ldots$$

The following integrals tends to infinity, the asymptotic growth is

$$\int_0^1 \prod_{k=1}^{n}(1+q^k)^k \; dq \sim \frac{3 * 2^{n(n+1)/2+1}}{n^3}$$

$$\int_0^1 \prod_{k=1}^{n}(1+q^k)^n \; dq \sim \frac{2^{n^2+2}}{n^3}$$

The following integral tends to zero if n tends to infinity, the asymptotics is

$$\int_0^1 \prod_{k=1}^{n}(1-q^k)^n \; dq \sim \frac{1}{n}$$

References:

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