Too many errors around coefficient C_1 in asymptotic of sequence A002720

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Main result: I found bug in program Mathematica!

Sequence A002720 in OEIS is number of *n* x *n* binary matrices with at most one 1 in each row and column or number of non-attacking placements of *k* rooks on an *n* x *n* board, summed over all *k* >= 0 (see [2], p.212).

Formula is simple

$$a_n = \sum_{k=0}^{n} \binom{n}{k}^2 k!$$

Values of this sequence are also special case of Laguerre polynomials

$$a_n = n! * \mathrm{LaguerreL}(n, -1)$$

Right asymptotic expansion is

$$a_n \sim \frac{n^{n+\frac{1}{4}}}{\sqrt{2}\, e^{n-2\sqrt{n}+\frac{1}{2}}} * \left(1 + \frac{31}{48\sqrt{n}} + \frac{553}{4608\, n} - \frac{222853}{3317760\, n\sqrt{n}} + \frac{9164693}{637009920\, n^2} + \cdots\right)$$

Therefore

$$\lim_{n\to\infty} \left(a_n * \frac{\sqrt{2}\, e^{n-2\sqrt{n}+\frac{1}{2}}}{n^{n+\frac{1}{4}}} - 1\right) * \sqrt{n} = \frac{31}{48}$$

In program Mathematica

```
Limit[(n!*LaguerreL[n,-1]/n^(n+1/4)*Sqrt[2]*E^(n-2*Sqrt[n]+1/2)-1)*Sqrt[n],n->Infinity]
```

But Mathematica wrong output (in versions 7 and 8) is 13/16

```
In[1]:= Limit[
          (n! * LaguerreL[n, -1] / n^(n + 1 / 4) * Sqrt[2] *
             E^(n - 2 * Sqrt[n] + 1 / 2) - 1) * Sqrt[n],
          n → Infinity]

Out[1]= 13
        --
        16
```

This is BUG !!!

But this is not all. Try same computing **numerically**, for example

```
N[Table[(n!*LaguerreL[n,-1]/n^(n+1/4)*Sqrt[2]*E^(n-2*Sqrt[n]+1/2)-1)*Sqrt[n],{n,1000,10000,1000}],20]
```

Or, with more terms and better precision (elapsed time was ~ 1 hour):

```
N[
 Table[
  (n! * LaguerreL[n, -1] / n^(n + 1 / 4) * Sqrt[2] *
      E^(n - 2 * Sqrt[n] + 1 / 2) - 1) * Sqrt[n],
  {n, 1 000 000, 10 000 000, 1 000 000}], 20]

{0.64595327485857865704, 0.64591815870538845793,
 0.64590259799022250717, 0.64589332088298149047,
 0.64588698941420000820, 0.64588231547803760347,
 0.64587868275613942973, 0.64587575441366589107,
 0.64587332876408306831, 0.64587127669377150813}
```

This is right !

31/48 = 0.645833
13/16 = 0.8125

Also with original sum are numerical results same (only computation with LaguerreL is faster)

```
N[
 Table[
  (Sum[Binomial[n, k]^2*k!, {k, 0, n}]/n^(n+1/4)*
      Sqrt[2]*E^(n-2*Sqrt[n]+1/2)-1)*Sqrt[n],
  {n, 10000, 100000, 10000}], 20]

{0.64702671774976334684, 0.64667856950124587690,
 0.64652396757356475342, 0.64643169930381838307,
 0.64636868636887773360, 0.64632214820918880927,
 0.64628596472380561470, 0.64625678910987046302,
 0.64623261647360573316, 0.64621216286263130538}
```

Similar numerical results produced also program Maple (unfortunately symbolic evaluation is not possible)

```
> for iter from 100000 by 100000 to 200000 do
  lag := seq(simplify(n!·LaguerreL(n, 0, -1),'LaguerreL'), n = iter..iter) :
  den := eval(n^(n + 1/4)/sqrt(2)/exp(n - 2*sqrt(n) + 1/2), n = iter) :
  print(evalf[50](((lag/den) - 1)·sqrt(iter)));od:
          0.64621216286263130537635272179788850602652001497785
          0.64610134521370374206073116052270670423741616765194
```

and following output is from program Maxima (only n=100 is possible)

```
(%i1) l(n):=(n!*gen_laguerre(n, 0, -1)/(n^(n+1/4)/sqrt(2)/exp(n-2*sqrt(n)+1/2))-1)*sqrt(n);
      ev(l(100),numer);
```

(%o1) $l(n) := \left(\dfrac{n!\,L_n^{(0)}(-1)}{\dfrac{\dfrac{n^{n+\frac{1}{4}}}{\sqrt{2}}}{\exp\left(n-2\sqrt{n}+\frac{1}{2}\right)}}-1\right)\sqrt{n}$

(%o2) 0.6571787769841

Under Mathematica I tested function **Hypergeometric1F1** yet, because

$$\text{Hypergeometric1F1}(-n, 1, -1) = \text{LaguerreL}(n, -1)$$

but results are same (probably same way in program):

```
Limit[
 (n!*Hypergeometric1F1[-n, 1, -1]/n^(n+1/4)*
     Sqrt[2]*E^(n-2*Sqrt[n]+1/2)-1)*Sqrt[n],
 n → Infinity]

13
──
16
```

BUG

```
N[
 Table[
  (n!*Hypergeometric1F1[-n, 1, -1]/n^(n+1/4)*
      Sqrt[2]*E^(n-2*Sqrt[n]+1/2)-1)*Sqrt[n],
  {n, 1000000, 10000000, 1000000}], 20]

{0.64595327485857865704, 0.64591815870538845793,
 0.64590259799022250717, 0.64589332088298149047,
 0.64588698941420000820, 0.64588231547803760347,
 0.64587868275613942973, 0.64587575441366589107,
 0.64587332876408306831, 0.64587127669377150813}
```

numerically OK

Now is time for historical note. Main term in the asymptotic expansion found Oskar Perron in 1921, see [3].

$$\Phi(\beta+n,\gamma;x)=\frac{\Gamma(\gamma)}{2\sqrt{\pi}}e^{\frac{1}{2}x}(xn)^{\frac{1}{4}-\frac{\gamma}{2}}e^{2\sqrt{xn}}\cdot\left[1+O\left(\frac{1}{\sqrt{n}}\right)\right]$$

In 1985 W. Van Assche published more detailed asymptotic expansion for Laguerre polynomials, but with **wrong term C_1** (see [4]) and in 2001 the same author published correction (see [5]) with **right term C_1**

$$L_n^{(-a)}(-z)=\frac{e^{-z/2}}{2\sqrt{\pi}}\frac{e^{2\sqrt{nz}}}{z^{1/4-a/2}\,n^{1/4+a/2}}\cdot\left(1+\left(\frac{3-12a^2+24(1-a)z+4z^2}{48\sqrt{z}}\right)\frac{1}{\sqrt{n}}+O\left(\frac{1}{n}\right)\right)$$

$$C_1=\frac{3-12a^2+24(1-a)*z+4z^2}{48\sqrt{z}}$$

In our special case is $z=1, a=0$ and therefore

$$C_1=\frac{31}{48}$$

For more details see [6], p. 3.

Recurrence for A002720 is

$$a_n = 2n\,a_{n-1} - (n-1)^2\,a_{n-2}$$

Program Asymptotics.m by Manuel Kauers generated asymptotic expansion from the recurrence.

```
In[3]:= Asymptotics[(-1 + n)^2 a[-2 + n] - 2 n a[-1 + n] + a[n], a[n], Order → 2]
```

$$\text{Out[3]=}\left\{e^{-2\sqrt{n}-n}\left(1+\frac{9164693}{637009920\,n^2}+\frac{222853}{3317760\,n^{3/2}}+\frac{553}{4608\,n}-\frac{31}{48\sqrt{n}}\right)n^{\frac{1}{4}+n},\ e^{2\sqrt{n}-n}\left(1+\frac{9164693}{637009920\,n^2}-\frac{222853}{3317760\,n^{3/2}}+\frac{553}{4608\,n}+\frac{31}{48\sqrt{n}}\right)n^{\frac{1}{4}+n}\right\}$$

Term C_1 is right! We must only select the dominant particular solution (in this case is dominant a second expression).

Remark: This program don't compute the multiplicative constant.

But this is still not all. I tried Maple package Algolib yet, with excellent function *equivalent* by Bruno Salvy. With help of Maple module gfun we can find differential equation for generating function from recurrence using *rectodiffeq* and then use *dsolve*. Recurrence relation is necessary transform for exponential generating function.

$$b_n = \frac{a_n}{n!}$$

New recurrence is

$$(n-1)\, b_{n-2} - 2n\, b_{n-1} + n\, b_n = 0$$

```
> # A002720 for Egf
  with(gfun) : eg := rectodiffeq( { (n − 1) b(n−2) −2 n b(n−1)
      + n b(n), b(1) =2, b(2) =7/2}, b(n), f(x) );
     simplify( dsolve( eg ) );
```

$$eg := \left\{ (x-2)\, f(x) + \left(x^2 - 2x + 1\right) \left(\frac{d}{dx} f(x) \right), f(0) = 1 \right\}$$

$$f(x) = -\frac{e^{-\frac{x}{x-1}}}{x-1}$$

This is right, exponential generating function is

$$\frac{1}{1-\mathrm{x}} * e^{\frac{\mathrm{x}}{1-\mathrm{x}}} = \sum_{n=0}^{\infty} b_n x^n = \sum_{n=0}^{\infty} \frac{a_n x^n}{n!}$$

Result from function *equivalent* we must multiple by n! (factorial). We obtain

$$> simplify\left(asympt\left(n! \cdot equivalent\left(-\frac{e^{-\frac{x}{x-1}}}{x-1}, x, n, 5 \right), n, 5 \right) \right) \text{assuming } n > 0;$$

$$\frac{1}{12} \left(6\sqrt{2}\, e^{-\frac{1}{2}} \sqrt{n} + 5\sqrt{2}\, e^{-\frac{1}{2}} + 12\, \mathrm{O}\left(\frac{1}{n^{3/4}} \right) n^{1/4} \right) e^{2\sqrt{n} - n}\, n^{-\frac{1}{4} + n}$$

Main term is right (and multiplicative constant is in closed form!), but **second term is wrong**, **C_1 is not 5/6**, but must be 31/48.

Note, that without success is in this case also excellent Maple package AsyRec by Doron Zeilberger. Problem is probably in Birkhoff -Trjitzinsky method for A002720.

```
> AsyC(N^2-2*(n + 2)*N + (n + 1)*(n + 1), n, N, 5, [2, 7], 1000);
                              FAIL
```

Finally one more remark. Another approximation of A002720 with **Bessel function** exists. But important is that this is not identity.

$$a_n \sim e^{-\frac{1}{2}} * \mathrm{BesselI}\left(0\,,2\sqrt{n}\right) * n!$$

Now we find C_1 with same system as in case of LaguerreL.

```
Limit[
  (E^(-1/2)*BesselI[0, 2*Sqrt[n]]*
      n!/(n^(n+1/4)/Sqrt[2]/E^(n-2*Sqrt[n]+1/2))-1)*
   Sqrt[n], n -> Infinity]
```

$$\frac{1}{16}$$

But **this is not bug**!
Numerically

```
N[
 Table[
  (E^(-1/2)*BesselI[0, 2*Sqrt[n]]*
      n!/(n^(n+1/4)/Sqrt[2]/E^(n-2*Sqrt[n]+1/2))-1)*
   Sqrt[n], {n, 1000000, 10000000, 1000000}], 20]
```

```
{0.062600925833894760366, 0.062571362362516080757,
 0.062558266047802008104, 0.062550459321562185933,
 0.062545131849898605468, 0.062541199325100346203,
 0.062538142998758594487, 0.062535679384222403356,
 0.062533638749176651031, 0.062531912441772792883}
```

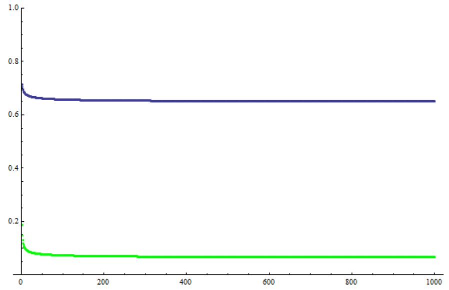
But 1/16 = 0.0625 and this **result is right** !

Asymptotic is not fully identical with A002720, only main term is identical.

$$a_n = n! * \mathrm{LaguerreL}(n,-1) \sim \frac{n^{n+\frac{1}{4}}}{\sqrt{2}\, e^{n-2\sqrt{n}+\frac{1}{2}}} * \left(1 + \frac{31}{48\sqrt{n}} + \cdots\right)$$

$$e^{-\frac{1}{2}} * \mathrm{BesselI}\left(0\,,2\sqrt{n}\right) * n! \sim \frac{n^{n+\frac{1}{4}}}{\sqrt{2}\, e^{n-2\sqrt{n}+\frac{1}{2}}} * \left(1 + \frac{1}{16\sqrt{n}} + \cdots\right)$$

```
asymp = n^(n+1/4)/Sqrt[2]/E^(n-2*Sqrt[n]+1/2);
Show[ListPlot[Table[(n!*LaguerreL[n, -1]/asymp-1)*Sqrt[n], {n, 1, 1000}],
  PlotRange → {0, 1}],
 ListPlot[Table[(E^(-1/2)*BesselI[0, 2*Sqrt[n]]*n!/asymp-1)*Sqrt[n],
   {n, 1, 1000}], PlotRange → {0, 1}, PlotStyle → Green]]
```



References:

[1] OEIS - The On-Line Encyclopedia of Integer Sequences

[2] V. Kotěšovec, Non-attacking chess pieces, 5th edition, 9.1.2012, p.212

[3] Oskar Perron, Über das Verhalten einer ausgearteten hypergeometrischen Reihe bei unbegrenztem Wachstum eines Parameters, Journal für die reine und angewandte Mathematik (1921), vol.151, p. 63-78

[4] W. Van Assche, Weighted zero distribution for polynomials orthogonal on an infinite interval, SIAM J. Math. Anal., 16 (1985), 1317–1334

[5] W. Van Assche, Erratum to "Weighted zero distribution for polynomials orthogonal on an infinite interval", SIAM J. Math. Anal., 32 (2001), 1169–1170.

[6] D. Borwein, Jonathan M. Borwein, Richard E. Crandall, Effective Laguerre asymptotics, 2008

[7] P. Flajolet a R. Sedgewick, Analytic combinatorics, p. 538-539

[8] Saber Elaydi, An Introduction to Difference Equations, 2005, p. 380-381

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